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12-1 ELL Support

Adding and Subtracting Matrices

Use the chart below to review vocabulary. These vocabulary words will help you complete this page.

Words	Explanations	Examples
Corresponding elements	Elements in the same position in a pair of matrices	$\begin{bmatrix} 2 & 1 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 6 & 7 \end{bmatrix}$ 9 and 6 are corresponding elements
Zero matrix	A matrix in which all elements are zero	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Additive inverse	A matrix in which each element is the opposite of the corresponding element in another matrix	$\begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 5 & -7 \end{bmatrix}$
Equal matrices	Matrices with the same dimensions and equal corresponding elements	$\begin{bmatrix} 0.25 & 7 \\ 6 & 4.5 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 7 \\ 6 & 4\frac{1}{2} \end{bmatrix}$

Use the vocabulary to fill in the blanks.

1. The matrices had the same corresponding elements, so they were **equal matrices**.
2. The sum of a matrix and its additive inverse is a **zero matrix**.
3. The **corresponding elements** are in the same position in a pair of matrices with equal dimensions.

Circle the correct answer.

4. What is the additive inverse of $\begin{bmatrix} -2 & 9 \\ -6 & 4 \end{bmatrix}$? $\begin{bmatrix} -2 & 9 \\ -6 & 4 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & -9 \\ 6 & -4 \end{bmatrix}$

5. Which matrix is equal to $\begin{bmatrix} 1.5 & 3 \\ 6 & 9.75 \end{bmatrix}$? $\begin{bmatrix} 3 & 9 \\ 1.5 & 6 \end{bmatrix}$ $\begin{bmatrix} 1\frac{1}{2} & 3 \\ 6 & 9\frac{3}{4} \end{bmatrix}$ $\begin{bmatrix} -1.5 & -3 \\ -6 & -9.75 \end{bmatrix}$

Find the sum.

6. $\begin{bmatrix} 4 & -6 \\ 9 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 4 & -6 \\ 9 & 1 \end{bmatrix}$

7. $\begin{bmatrix} -3 & 5 \\ 11 & -6 \end{bmatrix} + \begin{bmatrix} 3 & -5 \\ -11 & 6 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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12-1 Think About a Plan

Adding and Subtracting Matrices

Data Analysis Refer to the table.

a. Find the total number of people participating in each activity.
b. Find the difference between the numbers of males and females in each activity.
c. **Reasoning** In part (b), does the order of the matrices matter? Explain.

U.S. Participation in Selected Leisure Activities (millions)

Activity	Male	Female
Movies	59.2	65.4
Exercise Programs	54.3	59.0
Sports Events	40.5	31.1
Home Improvement	45.4	41.8

SOURCE: U.S. National Endowment for the Arts

1. Write matrices to show the information from the table.

$M = \begin{bmatrix} 59.2 \\ 54.3 \\ 40.5 \\ 45.4 \end{bmatrix}$ $F = \begin{bmatrix} 65.4 \\ 59.0 \\ 31.1 \\ 41.8 \end{bmatrix}$

2. Write a matrix equation to find the number of people, in millions, participating in each activity. $T =$ **$M + F$**

3. Solve the matrix equation. How many million people participate in each activity?

$T = \begin{bmatrix} 124.6 \\ 113.3 \\ 71.6 \\ 87.2 \end{bmatrix}$ Movies **124.6** Exercise Programs **113.3** Sports Events **71.6** Home Improvement **87.2**

4. Write a matrix equation to find the difference, in millions, between the number of males and females in each activity. $T =$ **$M - F$**

5. Solve the matrix equation. What is the difference, in millions, between the number of males and females in each activity?

$T = \begin{bmatrix} -6.2 \\ -4.7 \\ 9.4 \\ 3.6 \end{bmatrix}$ Movies **−6.2** Exercise Programs **−4.7** Sports Events **9.4** Home Improvement **3.6**

6. Does the order of the matrices matter? Explain. **Yes. Answers may vary. Sample: If you subtract matrix *M* from matrix *F*, then you will get the difference, in millions, between the number of females and males in each activity, which will change the signs of all elements in the matrix**.

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12-1 Practice *Form G*

Adding and Subtracting Matrices

Find each sum or difference.

1. $\begin{bmatrix} 3 & 2 \\ 8 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 4 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 \\ 12 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -4 \\ 1 & 2 \\ -7 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ -3 & 2 \\ -2 & 4 \end{bmatrix}$ $\begin{bmatrix} 3 & -9 \\ 4 & 0 \\ -5 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 3 \\ 9 & -1 \\ 2 & 4 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 6 & 2 \\ 10 & -1 \\ 5 & 6 \\ -2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0.5 & -0.1 \\ 1.2 & 2.3 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & -1.4 \end{bmatrix}$ $\begin{bmatrix} 0.3 & -0.2 \\ 0.8 & 3.7 \end{bmatrix}$

Solve each matrix equation.

5. $X - \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 12 \\ 3 & 2 \end{bmatrix}$ $\begin{bmatrix} 8 & 11 \\ 13 & 14 \\ 4 & 11 \end{bmatrix}$

6. $X + \begin{bmatrix} 20 & -9 & -3 \\ 19 & -2 & -5 \\ -1 & 0 & -8 \end{bmatrix} = \begin{bmatrix} -7 & 92 & -5 \\ 0 & 91 & -6 \\ -9 & -1 & 12 \end{bmatrix}$ $\begin{bmatrix} -27 & 101 & -2 \\ -19 & 93 & -1 \\ -8 & -1 & 20 \end{bmatrix}$

7. $\begin{bmatrix} -2 & -3 \\ 2 & 2 \end{bmatrix} = X - \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ $\begin{bmatrix} -1 & -4 \\ 0 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & -3 & 4 \end{bmatrix} - X$ $\begin{bmatrix} 0 & -4 & 3 \\ -4 & -2 & 5 \end{bmatrix}$

Find each sum.

9. $\begin{bmatrix} 5 & -2 & 1 \\ 0 & -3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 2 & -1 \\ 0 & 3 & -4 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -8 & 8 \\ 9 & -9 \end{bmatrix}$ $\begin{bmatrix} -8 & 8 \\ 9 & -9 \end{bmatrix}$

Find the value of each variable.

11. $\begin{bmatrix} 8 & 6 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3a - 1 & 2a \\ 5b + 3 & a + 3b \end{bmatrix}$
$a = 3; b = -1$

12. $\begin{bmatrix} 4 & -3 \\ 7 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} p & q \\ 4 & r \end{bmatrix}$
$p = 2; q = -3; r = 3$

Find each matrix sum or difference if possible. If not possible, explain.

$P = \begin{bmatrix} 0 & 2 & 4 \\ 9 & 8 & 2 \end{bmatrix}$ $Q = \begin{bmatrix} -2 & -4 & 1 \\ 9 & 7 & 0 \end{bmatrix}$ $R = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 5 \\ 0 & -6 & 1 \end{bmatrix}$ $S = \begin{bmatrix} 0 & -6 & 7 \\ 3 & 8 & 2 \\ 0 & -1 & 5 \end{bmatrix}$

13. $P + Q$ $\begin{bmatrix} -2 & -2 & 5 \\ 18 & 15 & 2 \end{bmatrix}$

14. $S - R$ $\begin{bmatrix} -4 & -5 & 7 \\ 1 & 5 & -3 \\ 0 & 5 & 4 \end{bmatrix}$

15. $Q + R$
Not possible; they do not have the same dimensions.

16. $Q - P$ $\begin{bmatrix} -2 & -6 & -3 \\ 0 & -1 & -2 \end{bmatrix}$

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12-1 Practice (continued) *Form G*

Adding and Subtracting Matrices

17. The table shows the number of males and females in four clubs at a high school for two school years.

Club Membership

	1961–1962		2009–2010	
	Males	Females	Males	Females
Beta	37	23	56	58
Spanish	0	93	76	82
Chess	87	0	102	34
Library	6	18	27	29

$A = \begin{bmatrix} 37 \\ 0 \\ 87 \\ 6 \end{bmatrix}; B = \begin{bmatrix} 23 \\ 93 \\ 0 \\ 18 \end{bmatrix};$ $C = \begin{bmatrix} 56 \\ 76 \\ 102 \\ 27 \end{bmatrix}; D = \begin{bmatrix} 58 \\ 82 \\ 34 \\ 29 \end{bmatrix}$

a. Write four 4×1 matrices, A, B, C, and D, to represent the male and female club membership for 1961–1962 and 2009–2010.
b. Write and solve a matrix equation to find matrix X, the total number of members in each club for 1961–1962. $A + B = X; X = \begin{bmatrix} 60 \\ 93 \\ 87 \\ 24 \end{bmatrix}$
c. Did the total number of female club members increase or decrease between the two school years? By what amount? **increase; 69**

18. **Reasoning** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, and $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
If $a_{11} \cdot b_{11} = -16$ and $a_{11} > 0$, what is the value of b_{11}? **−4**

19. The table shows the time each member of two relay teams took to complete her leg of a relay race. Team II won the race by 2 s. How many seconds did Trina take to run her leg of the race? **24 s**

Relay Race Results

	Team I		Team II	
Leg	Name	Time (s)	Name	Time (s)
1	Ali	22	Lea	23
2	Bryn	25	Niki	22
3	Mai	23	Trina	
4	Tara	21	Sara	20

Writing Determine whether the two matrices in each pair are equal. Explain.

20. $\begin{bmatrix} 2 \\ \sqrt{9} \\ 16 \end{bmatrix}; \begin{bmatrix} \frac{4}{2} & 3 & 4^2 \end{bmatrix}$ **Not equal; the dimensions of the two matrices are different.**

21. $\begin{bmatrix} 2(3) & 3(1.5) \\ 7 & \frac{10}{2} \end{bmatrix}; \begin{bmatrix} 6 & 4.5 \\ 7 & 5 \end{bmatrix}$ **Equal; the dimensions and the corresponding elements of the two matrices are equal.**

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12-1 Practice

Form K

Adding and Subtracting Matrices

Find each sum or difference.

To start, add or subtract corresponding elements.

1. $\begin{bmatrix} 2 & 4 \\ 5 & -7 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 3 & 5 \end{bmatrix}$

$\begin{bmatrix} 2 + (-4) & 4 + (-1) \\ 5 + 3 & -7 + 5 \end{bmatrix}$

$\begin{bmatrix} -2 & 3 \\ 8 & -2 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 3 \\ 8 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 7 \\ 4 & -3 \end{bmatrix}$

$\begin{bmatrix} 4 & -4 \\ 4 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -3 & 7 & 1 \\ 4 & 3 & -2 \end{bmatrix} 1 \begin{bmatrix} 5 & 2 & -6 \\ -4 & 6 & 9 \end{bmatrix}$

$\begin{bmatrix} 2 & 9 & -5 \\ 0 & 9 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 2 & -2 & 6 \\ -4 & 3 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 1 \\ 2 & 7 & 4 \end{bmatrix}$

$\begin{bmatrix} -3 & -5 & 5 \\ -6 & -4 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 9 & -6 \\ -2 & 5 \\ 8 & -1 \end{bmatrix} + \begin{bmatrix} -4 & 7 \\ 5 & 3 \\ 1 & 6 \end{bmatrix}$

$\begin{bmatrix} 5 & 1 \\ 3 & 8 \\ 9 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 4 \\ 8 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 2 \end{bmatrix}$

$\begin{bmatrix} -4 & -5 \\ 5 & 3 \end{bmatrix}$

Solve each matrix equation.

To start, use the Addition Property of Equality to isolate the variable matrix.

7. $\begin{bmatrix} 7 & -1 \\ 3 & 5 \end{bmatrix} + X = \begin{bmatrix} 4 & 5 \\ 8 & 2 \end{bmatrix}$

$X = \begin{bmatrix} 4 - 7 & 5 - (-1) \\ 8 - 3 & 2 - 5 \end{bmatrix}$

$X = \begin{bmatrix} -3 & 6 \\ 5 & -3 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix} - X = \begin{bmatrix} -5 & 1 \\ 3 & 4 \end{bmatrix}$

$X = \begin{bmatrix} 7 & 6 \\ 6 & -7 \end{bmatrix}$

9. $X - \begin{bmatrix} 2 & 8 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 4 & 2 \end{bmatrix}$

$X = \begin{bmatrix} 5 & 2 \\ 3 & 7 \end{bmatrix}$

10. Error Analysis Maria added $\begin{bmatrix} 5 & 9 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} -2 & -5 \\ 6 & 3 \end{bmatrix}$ and found a sum of $\begin{bmatrix} 0 & 7 \\ 4 & 3 \end{bmatrix}$. What error did Maria make, and what is the correct sum? **Maria added corresponding elements in the two matrices but did not put them in the correct place in the sum; $\begin{bmatrix} 3 & 4 \\ 7 & 0 \end{bmatrix}$.**

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12-1 Practice (continued)

Form K

Adding and Subtracting Matrices

Find each sum.

11. $\begin{bmatrix} 3 & -1 \\ -5 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 5 & -8 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} -4 & 9 \\ -7 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -4 & 9 \\ -7 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 7 & 4 \\ 5 & -2 \end{bmatrix} + \begin{bmatrix} -7 & -4 \\ -5 & 2 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Find the value of each variable.

14. $\begin{bmatrix} 3 & 4 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 5 & -5 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} 8 & x \\ y & z \end{bmatrix}$

$x = 4 + (-5) = \boxed{-1}$

$y = -1 + 7 = \boxed{6}$

$z = 6 + (-3) = \boxed{3}$

15. $\begin{bmatrix} 2x & 7 \\ -4 & 3y - 1 \end{bmatrix} = \begin{bmatrix} 18 & 7 \\ -4 & 8 \end{bmatrix}$

$x = 9$; $y = 3$

16. $\begin{bmatrix} -8 & 3 & -5 \\ 1 & -7 & -6 \end{bmatrix} = \begin{bmatrix} 5x + 2 & 3 & 2 - y \\ 1 & 2z + 1 & -6 \end{bmatrix}$

$x = -2$; $y = 7$; $z = -4$

17. $\begin{bmatrix} 13 & 4b + 3 \\ -5a & -6 \end{bmatrix} = \begin{bmatrix} 3c + 1 & 11 \\ -25 & -2c + 2 \end{bmatrix}$

$a = 5$; $b = 2$; $c = 4$

18. Writing Describe the Commutative and Associative Properties of Matrix Addition. How are these properties similar to the Commutative and Associative Properties of Real-Number Addition? **The Commutative Property of Matrix Addition states that matrices can be added in any order and the sum will remain the same. The Associative Property of Matrix Addition states that matrices can be grouped in any way and the sum will remain the same. The properties of matrix addition are the same as those of real numbers, except the properties apply to the elements of a matrix.**

19. Reasoning Is it possible to find the value of x in the following equation? Why or why not?

$$\begin{bmatrix} 2x & -1 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 5x - 4 & -1 \\ 4 & -5 \end{bmatrix}$$

Yes; the variable appears in the same pair of corresponding elements. Solving the equation $2x = 5x - 4$ gives the answer $x = \frac{4}{3}$.

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12-1 Standardized Test Prep

Adding and Subtracting Matrices

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. What matrix is equal to the difference $\begin{bmatrix} 5 & 9 & -3 \\ 6 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 2 \\ 0 & 3 & 5 \end{bmatrix}$? **C**

(A) $\begin{bmatrix} -1 & -5 & -5 \\ -6 & -5 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -5 & 5 \\ -6 & 5 & 4 \end{bmatrix}$ (C) $\begin{bmatrix} -1 & 5 & -5 \\ 6 & -5 & -4 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 5 & 5 \\ 6 & 5 & 4 \end{bmatrix}$

2. Which matrix is equivalent to X in the equation $\begin{bmatrix} 4 & 0 \\ 1 & -2 \end{bmatrix} + X = \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}$ **F**

(F) $\begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix}$ (G) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (H) $\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ (I) $\begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$

3. Which matrix is equivalent to P in the equation $\begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} - P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ **D**

(A) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} -7 & -8 \\ -9 & -10 \\ -11 & -12 \end{bmatrix}$ (D) $\begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$

4. Let $R + S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. If $R = \begin{bmatrix} -3 & 2 & 9 \\ 7 & 6 & -4 \end{bmatrix}$, which matrix is equivalent to S? **H**

(F) $\begin{bmatrix} -3 & 2 & 9 \\ 7 & 6 & -4 \end{bmatrix}$ (G) $\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ (H) $\begin{bmatrix} 3 & -2 & -9 \\ -7 & -6 & 4 \end{bmatrix}$ (I) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Short Response

5. If $\begin{bmatrix} 8 & 2x - 1 \\ 2y + 1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -7 \\ y & -x \end{bmatrix}$, what values of x and y make the equation true? Show your work.

[2] $2x - 1 = -7$ $2y + 1 = y$ $3 = -x$
$2x = -6$ $y = -1$ $-3 = x$
$x = -3$
The solution is $x = -3$, $y = -1$.
[1] partially incorrect or incomplete work shown
[0] incorrect answers and no work shown OR no answers given

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12-1 Enrichment

Adding and Subtracting Matrices

Transpose is another operation that you can apply to a matrix. You form the transpose of a matrix by switching the rows for columns. For example, matrix A^T below is the transpose of matrix A.

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 5 & -6 & 2 \\ 3 & -3 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -1 & 5 & 3 \\ 3 & -6 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

1. Write the transpose of matrix $R = \begin{bmatrix} 5 & 11 \\ -4 & 1 \end{bmatrix}$. $R^T = \begin{bmatrix} 5 & -4 \\ 11 & 1 \end{bmatrix}$

In 1934, an Indian student named S. P. Sundaram constructed an interesting matrix. The first row in his matrix was 4, 7, 10, 13, 16, and so on. He then created the first column by transposing this row.

$$\begin{bmatrix} 4 & 7 & 10 & 13 & \dots \\ 7 & & & & \\ 10 & & & & \\ 13 & & & & \\ \vdots & & & & \end{bmatrix}$$

2. What is the pattern used to generate each number in the first row? **add 3 to the element to the left**

3. Each successive row increases by the next odd number, so the numbers in row two grow by adding 5, the numbers in row three grow by adding 7, and so on. Fill in the remaining elements of this matrix.

$$\begin{bmatrix} 4 & 7 & 10 & 13 \\ 7 & 12 & 17 & 22 \\ 10 & 17 & 24 & 31 \\ 13 & 22 & 31 & 40 \end{bmatrix}$$

This matrix is called the sieve of Sundaram and is used to find all of the prime numbers up to a certain integer.

4. Choose a number in the sieve of Sundaram. Double it and add one. Is your final number prime? Try several more numbers. **Answers may vary. Sample: 7, 2(7) + 1 = 15; for the chosen number, the final number is not prime.**

5. Choose a number that is not in the sieve of Sundaram. Double it and add one. Try several numbers. Are your final numbers prime? **Answers may vary. Sample: 11, 2(11) + 1 = 23; yes, the final numbers are all prime for the chosen numbers.**

6. Explain how the sieve of Sundaram helps determine if a number will be prime or not. **Answers may vary. Sample: If a number *n* does not appear in the sieve of Sundaram, then $2n + 1$ is prime.**

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12-1 Reteaching
Adding and Subtracting Matrices

A matrix is like a table without the row and column labels.

To add or subtract matrices of the same size, combine the *corresponding elements*, the numbers in the same position in each matrix. The sum or difference of matrices will have the same *dimensions*, the number of rows and columns, as the matrices you combined.

To help you keep track of your work, draw lines between the rows and columns and cross off elements as you combine them.

Voting Records

Table

	Yes	No
Brown	29	51
Montoya	45	35

Matrix $\begin{bmatrix} 29 & 51 \\ 45 & 35 \end{bmatrix}$

Problem

What is $\begin{bmatrix} -3 & 5 \\ 9 & -2 \end{bmatrix} + \begin{bmatrix} 7 & -1 \\ 8 & -4 \end{bmatrix}$?

$\begin{bmatrix} -3 & 5 \\ 9 & -2 \end{bmatrix} + \begin{bmatrix} 7 & -1 \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$ The sum will be a 2 × 2 matrix.

$\begin{bmatrix} \cancel{-3} & 5 \\ 9 & -2 \end{bmatrix} + \begin{bmatrix} \cancel{7} & -1 \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 4 & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$ Add row 1, column 1 elements. Cross them off.

$\begin{bmatrix} \cancel{-3} & \cancel{5} \\ 9 & -2 \end{bmatrix} + \begin{bmatrix} \cancel{7} & \cancel{-1} \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ \blacksquare & \blacksquare \end{bmatrix}$ Add row 1, column 2 elements. Cross them off.

$\begin{bmatrix} \cancel{-3} & \cancel{5} \\ \cancel{9} & \cancel{-2} \end{bmatrix} + \begin{bmatrix} \cancel{7} & \cancel{-1} \\ \cancel{8} & \cancel{-4} \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 17 & -6 \end{bmatrix}$ Repeat for the remaining matrix elements.

So, $\begin{bmatrix} -3 & 5 \\ 9 & -2 \end{bmatrix} + \begin{bmatrix} 7 & -1 \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 17 & -6 \end{bmatrix}$.

Exercises

Find each sum or difference.

1. $\begin{bmatrix} -3 & 8 \\ 9 & -2 \end{bmatrix} + \begin{bmatrix} 1 & -5 \\ 5 & 0 \end{bmatrix}$ $\begin{bmatrix} -2 & 3 \\ 14 & -2 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 5 \\ 3 & 1 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & -5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -2 \\ 0 & -6 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ -1 & -8 \end{bmatrix}$ $\begin{bmatrix} -5 & 1 \\ 1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} -9 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -7 & -3 & -4 \\ 8 & -7 & -9 \end{bmatrix}$ $\begin{bmatrix} -16 & -1 & -4 \\ 7 & -7 & -6 \end{bmatrix}$

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12-1 Reteaching (continued)
Adding and Subtracting Matrices

Solving matrix equations is like solving other kinds of algebraic equations. Isolate the variable on one side of the equal sign, and then simplify the expression on the other side.

Problem

If $X = \begin{bmatrix} 5 & 3 & -1 & 8 \\ 2 & -4 & 9 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 6 & 7 & 2 & -2 \\ 8 & 0 & -1 & 4 \end{bmatrix}$, and $X + Z = Y$, what is Z?

$X + Z = Y$ Write the equation.

$Z = Y - X$ Subtract X from both sides to isolate Z.

$Z = \begin{bmatrix} 6 & 7 & 2 & -2 \\ 8 & 0 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 3 & -1 & 8 \\ 2 & -4 & 9 & 0 \end{bmatrix}$ Substitute for X and Y.

$Z = \begin{bmatrix} 6-5 & 7-3 & 2-(-1) & -2-8 \\ 8-2 & 0-(-4) & -1-9 & 4-0 \end{bmatrix}$ Subtract corresponding elements.

$Z = \begin{bmatrix} 1 & 4 & 3 & -10 \\ 6 & 4 & -10 & 4 \end{bmatrix}$ Simplify.

Corresponding elements of equivalent matrices are equal. You can use this fact to find the value of unknown matrix elements.

Problem

What value of x makes $\begin{bmatrix} 3x-1 & 6 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$ a true statement?

$3x - 1 = 5$ Corresponding elements of equivalent matrices are equal.

$3x = 6$ Solve for x.

$x = 2$

Exercises

Solve each matrix equation.

5. $\begin{bmatrix} 1.5 & 0.5 \\ -2.5 & 2.5 \end{bmatrix} - A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\begin{bmatrix} -1.5 & -3.5 \\ -7.5 & -3.5 \end{bmatrix}$

6. $C + \begin{bmatrix} -1 & -4 \\ 0 & 5 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ 0 & -4 \\ -1 & 5 \end{bmatrix}$ $\begin{bmatrix} -7 & 2 \\ 0 & -9 \\ -10 & 5 \end{bmatrix}$

Find the value of each variable.

7. $\begin{bmatrix} 4 & 2 \\ -4 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 3a \\ -4 & 9 \end{bmatrix}$ $\frac{2}{3}$

8. $\begin{bmatrix} -3 & f & 2.4 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3g & 7 & 2.4 \\ 3 & 0 & h+3 \end{bmatrix}$ $f = 7; g = -1; h = -2$

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12-2 ELL Support
Matrix Multiplication

Kerri is learning how to multiply matrices. She wrote the steps to multiply $A = \begin{bmatrix} 3 & 7 \\ 1 & -2 \end{bmatrix}$ by $B = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$ on note cards, but the cards got mixed up.

Multiply the elements in the first row of A by the elements in the second column of B. Place the sum in the first row, second column of AB.

Multiply the elements in the second row of A by the elements in the second column of B. Place the sum in the second row, second column of AB.

Multiply the elements in the second row of A by the elements in the first column of B. Place the sum in the second row, first column of AB.

Multiply the elements in the first row of A by the elements in the first column of B. Place the sum in the first row, first column of AB.

Use the note cards to write the steps in order.

1. First, multiply the elements in the first row of A by the elements in the first column of B. Place the sum in the first row, first column of AB.
2. Second, multiply the elements in the first row of A by the elements in the second column of B. Place the sum in the first row, second column of AB.
3. Then, multiply the elements in the second row of A by the elements in the first column of B. Place the sum in the second row, first column of AB.
4. Finally, multiply the elements in the second row of A by the elements in the second column of B. Place the sum in the second row, second column of AB.

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12-2 Think About a Plan
Matrix Multiplication

Sport Two teams are competing in a track meet. Points for individual events are awarded as follows: 5 points for first place, 3 points for second place, and 1 point for third place. Points for team relays are awarded as follows: 5 points for first place and no points for second place.

a. Use matrix operations to determine the score in the track meet.

b. Who would win if the scoring was changed to 5 points for first place, 2 points for second place, and 1 point for third place in each individual event with relay scoring remaining 5 points for first place?

	Individual Events			Relays	
Team	First	Second	Third	First	Second
West River	8	5	2	8	5
River's Edge	6	9	12	6	9

Know

1. The number of each place for each school and the point value of each place.

Need

2. To solve the problem I need to: write the number of wins and the point values as matrices, then multiply the matrices.

Plan

3. Write the number of wins as a 2 × 5 matrix and the original and alternate point values as 5 × 1 matrices. $\begin{bmatrix} 8 & 5 & 2 & 8 & 5 \\ 6 & 9 & 12 & 6 & 9 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 3 \\ 1 \\ 5 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}$

4. Use matrix multiplication to find the original total team scores and the alternate total team scores for the track meet.

$\begin{bmatrix} 8 & 5 & 2 & 8 & 5 \\ 6 & 9 & 12 & 6 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 97 \\ 99 \end{bmatrix}$ $\begin{bmatrix} 8 & 5 & 2 & 8 & 5 \\ 6 & 9 & 12 & 6 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 92 \\ 90 \end{bmatrix}$

5. What was the score in the track meet? West River: 97, River's Edge: 99

6. Who would win if the scoring were changed? West River

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12-2 Practice

Matrix Multiplication *Form G*

Use matrices A, B, C, and D. Find each product, sum, or difference.

$A = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 \\ -2 & 1 \\ -1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & 4 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1. $2D$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

2. $0.2B$ $\begin{bmatrix} 0 & 0.4 \\ -0.4 & 0.2 \\ -0.2 & 0 \end{bmatrix}$

3. $\frac{1}{4}C$ $\begin{bmatrix} \frac{3}{4} & -\frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$

4. DC $\begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & 4 \end{bmatrix}$

5. BD $\begin{bmatrix} 0 & 2 \\ -2 & 1 \\ -1 & 0 \end{bmatrix}$

6. $2A + 4D$ $\begin{bmatrix} 6 & -2 \\ 6 & 0 \end{bmatrix}$

7. $5D - A$ $\begin{bmatrix} 4 & 1 \\ -3 & 7 \end{bmatrix}$

8. $3D + A$ $\begin{bmatrix} 4 & -1 \\ 3 & 1 \end{bmatrix}$

9. $3C - 2DC$ $\begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & 4 \end{bmatrix}$

Solve each matrix equation. Check your answers.

10. $2\begin{bmatrix} 0 & 1 \\ 3 & -4 \end{bmatrix} - 3X = \begin{bmatrix} 9 & -6 \\ 1 & -2 \end{bmatrix}$ $\begin{bmatrix} -3 & \frac{8}{3} \\ \frac{5}{3} & -2 \end{bmatrix}$

11. $\frac{1}{2}X + \begin{bmatrix} 5 & -1 \\ 0 & \frac{2}{3} \end{bmatrix} = 2\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 2 \\ 4 & \frac{20}{3} \end{bmatrix}$

Find each product.

12. $\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -3 & 1 \end{bmatrix}$ $\begin{bmatrix} 3 & 7 \\ -9 & 23 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$ $\begin{bmatrix} 20 & 12 \\ -1 & 6 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ $[41]$

15. $\begin{bmatrix} 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & -1 \\ 0 & 6 \end{bmatrix}$ $[0 \;\; 41]$

16. $\begin{bmatrix} 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 0 \\ 6 & 0 \end{bmatrix}$ $[41 \;\; 0]$

17. $\begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 0 \\ 6 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 41 & 0 \end{bmatrix}$

18. $\begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}$ $\begin{bmatrix} 25 & 0 \\ 0 & 16 \end{bmatrix}$

19. $\begin{bmatrix} 4 & 3 \\ 9 & 7 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 9 & 4 \end{bmatrix}$ $\begin{bmatrix} 51 & 24 \\ 117 & 55 \end{bmatrix}$

20. $\begin{bmatrix} 3 & 1 \\ -2 & 4 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} -3 & -1 \\ 2 & -4 \\ 1 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 4 & 1 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 27 \\ -13 \end{bmatrix}$

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12-2 Practice (continued)

Matrix Multiplication *Form G*

22. A carpenter builds three boxes. One box uses 12 nails. The second box uses 6 nails and 6 screws. The third box uses 8 screws and 2 hinges. Nails cost \$.04 each, screws cost \$.06 each, and hinges cost \$.12 each.

a. Write a matrix to show the number of each type of hardware in each box.

b. Write a matrix to show the cost of each type of hardware.

c. Find the matrix showing the cost of hardware for each box.

$\begin{bmatrix} 12 & 0 & 0 \\ 6 & 6 & 0 \\ 0 & 8 & 2 \end{bmatrix}$; $\begin{bmatrix} 0.04 \\ 0.06 \\ 0.12 \end{bmatrix}$; $\begin{bmatrix} 0.48 \\ 0.60 \\ 0.72 \end{bmatrix}$

Determine whether the product exists.

$P = \begin{bmatrix} 4 & -5 \\ 0 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ $R = \begin{bmatrix} -3 & 2 \end{bmatrix}$ $S = \begin{bmatrix} 0 & -1 \\ 4 & 6 \end{bmatrix}$

23. SP yes **24.** QS no **25.** PR no **26.** QR yes

27. A rugby game consists of two 40-min halves. In rugby, a try (T) is 5 points, a conversion kick (C) is 2 points, a penalty kick (PK) is 3 points, and a drop goal (DG) is 3 points.

a. Use matrix operations to determine the score in a game between the Austin Huns and the Dallas Harlequins. Austin 33, Dallas 29

Austin Huns vs. Dallas Harlequins

	First Half				Second Half			
Team	T	C	PK	DG	T	C	PK	DG
Austin	2	2	1	0	2	0	2	0
Dallas	1	0	3	0	2	1	0	1

b. Many years ago, a try was worth only 4 points and a conversion was worth 3 points. If the second half were scored by the old rules, which team would win the game? Austin

28. Reasoning Real-number multiplication is commutative. Is the same true for matrix multiplication? Explain your reasoning. No; answers may vary. Sample: Changing the order of the factors changes the product. For example, if A and B are 2×2 matrices, $(ab)_{11} = a_{11}b_{11} + a_{12}b_{21}$, but $(ba)_{11} = a_{11}b_{11} + a_{21}b_{12}$.

29. Error Analysis A student says $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is the multiplicative identity for a 2×2 matrix. Do you agree? If not, what is the correct matrix? No; answers may vary. Sample: The multiplicative identity for a 2×2 matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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12-2 Practice

Matrix Multiplication *Form K*

Let $A = \begin{bmatrix} 2 & -7 \\ -5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 4 \\ 1 & -11 \end{bmatrix}$. Find each product and each sum.

1. $2A$

$2\begin{bmatrix} 2 & -7 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(-7) \\ 2(-5) & 2(3) \end{bmatrix}$

$\begin{bmatrix} 4 & -14 \\ -10 & 6 \end{bmatrix}$

2. $5B$ $\begin{bmatrix} -30 & 20 \\ 5 & -55 \end{bmatrix}$

3. $3A + 4B$ $\begin{bmatrix} -18 & -5 \\ -11 & -35 \end{bmatrix}$

Solve each matrix equation.

To start, use the Subtraction Property of Equality to isolate the variable matrix.

4. $2X + \begin{bmatrix} 4 & -5 \\ 1 & -12 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ -7 & -2 \end{bmatrix}$

$2X = \begin{bmatrix} 10 & 1 \\ -7 & -2 \end{bmatrix} - \begin{bmatrix} 4 & -5 \\ 1 & -12 \end{bmatrix}$

$X = \begin{bmatrix} 3 & 3 \\ -4 & 5 \end{bmatrix}$

5. $4\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} - \frac{1}{2}X = \begin{bmatrix} 3 & 9 \\ -2 & 6 \end{bmatrix}$

$X = \begin{bmatrix} -22 & 14 \\ 28 & -20 \end{bmatrix}$

6. $5\begin{bmatrix} 1 & 5 \\ 4 & 3 \end{bmatrix} + 3X = \begin{bmatrix} 14 & 22 \\ 8 & 18 \end{bmatrix}$

$X = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$

7. $\frac{1}{4}X + \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -8 & 16 \end{bmatrix}$

$X = \begin{bmatrix} 20 & 24 \\ -12 & 36 \end{bmatrix}$

8. Open-Ended Write an example to demonstrate that the Associative Property applies to scalar multiplication.

Answers may vary. Sample: $(3 \cdot 2) \cdot \begin{bmatrix} 5 & 8 \\ 2 & 3 \end{bmatrix} = 6 \cdot \begin{bmatrix} 5 & 8 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 30 & 48 \\ 12 & 18 \end{bmatrix}$;
$3 \cdot \left(2 \cdot \begin{bmatrix} 5 & 8 \\ 2 & 3 \end{bmatrix}\right) = 3 \cdot \begin{bmatrix} 10 & 16 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 30 & 48 \\ 12 & 18 \end{bmatrix}$

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12-2 Practice (continued)

Matrix Multiplication *Form K*

Find each product.

To start, find the element in the first row and first column of the product matrix.

9. $\begin{bmatrix} 4 & -2 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 1 & -5 \end{bmatrix}$

$4(3) + (-2)(1) = 10 \rightarrow \begin{bmatrix} 10 & \\ & \end{bmatrix}$

$4(6) + (-2)(-5) = 34 \rightarrow \begin{bmatrix} 10 & 34 \\ & \end{bmatrix}$

$(-3)(3) + 7(1) = -2 \rightarrow \begin{bmatrix} 10 & 34 \\ -2 & \end{bmatrix}$

$(-3)(6) + 7(-5) = -53 \rightarrow \begin{bmatrix} \boxed{10} & \boxed{34} \\ \boxed{-2} & \boxed{-53} \end{bmatrix}$

10. $\begin{bmatrix} 5 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$ $\begin{bmatrix} 7 & -3 \\ -4 & 6 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 4 & 7 \end{bmatrix}$ $[18 \;\; 41]$

12. $\begin{bmatrix} 6 & -2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 8 & 3 \end{bmatrix}$ $\begin{bmatrix} -10 & 24 \\ -3 & -15 \end{bmatrix}$

Determine whether the product exists.

$A = \begin{bmatrix} 2 & 0 \\ -6 & 9 \end{bmatrix}$ $B = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ $C = \begin{bmatrix} 1 & -3 \\ 13 & -5 \end{bmatrix}$ $D = \begin{bmatrix} -7 & 5 \end{bmatrix}$

13. AC yes **14.** BA no **15.** DC yes **16.** BD yes

17. The table below shows the number of small, medium, large, and extra-large drinks sold at two snack stands in an hour. The small drinks cost \$1.00, the medium drinks cost \$1.50, the large drinks cost \$2.00, and the extra-large drinks cost \$2.50. Using matrix multiplication, what was the sales total for each snack stand? Stand 1: \$70.50; Stand 2: \$87.00

	Small	Medium	Large	Extra Large
Stand 1	16	8	10	9
Stand 2	6	12	9	18

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12-2 Standardized Test Prep
Matrix Multiplication

Multiple Choice

For Exercises 1–3, choose the correct letter.

1. Which matrix is equivalent to $-2\begin{bmatrix} 1 & 5 & -3 \\ 0 & 2 & 4 \\ 7 & -2 & 0 \end{bmatrix}$? A

(A) $\begin{bmatrix} -2 & -10 & 6 \\ 0 & -4 & -8 \\ -14 & 4 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} -2 & -10 & 6 \\ 0 & 2 & 4 \\ 7 & -2 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & -4 & -8 \\ 7 & -2 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 3 & -5 \\ -2 & 0 & 2 \\ 5 & -4 & -2 \end{bmatrix}$

2. What is the product $\begin{bmatrix} 6 & -1 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \end{bmatrix}$? H

(F) $\begin{bmatrix} 18 & -3 \\ -18 & -54 \end{bmatrix}$ (G) $\begin{bmatrix} 24 & -45 \end{bmatrix}$ (H) $\begin{bmatrix} 24 \\ -45 \end{bmatrix}$ (I) $\begin{bmatrix} 15 & 36 \\ -30 & -72 \end{bmatrix}$

3. Which matrix is the solution of $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \end{bmatrix} - 2X = \begin{bmatrix} 4 & 5 & 6 \\ 6 & 5 & 4 \end{bmatrix}$? D

(A) $\begin{bmatrix} 3 & 6 & 4 \\ 4 & 5 & 5 \end{bmatrix}$ (C) $\begin{bmatrix} \frac{3}{2} & 3 & 2 \\ 2 & \frac{5}{2} & \frac{5}{2} \end{bmatrix}$

(B) $\begin{bmatrix} -6 & -12 & -8 \\ -8 & -10 & -10 \end{bmatrix}$ (D) $\begin{bmatrix} -\frac{3}{2} & -3 & -2 \\ -2 & -\frac{5}{2} & -\frac{5}{2} \end{bmatrix}$

$T = \begin{bmatrix} 20 & 50 & 10 \\ 15 & 30 & 5 \\ 25 & 100 & 50 \end{bmatrix}\begin{bmatrix} 1.20 \\ 1.50 \\ 0.80 \end{bmatrix} = \begin{bmatrix} 20(1.20) + 50(1.50) + 10(0.80) \\ 15(1.20) + 30(1.50) + 5(0.80) \\ 25(1.20) + 100(1.50) + 50(0.80) \end{bmatrix} = \begin{bmatrix} 107 \\ 67 \\ 220 \end{bmatrix}$

Extended Response

$107 for Bath 1, $67 for Bath 2, $220 for the kitchen, so $394 total.

4. The table shows the number of tiles used in a house. Blue tiles cost $1.20 each, white cost $1.50 each, and green cost $.80 each. Write and solve a matrix equation to find the total cost of the tile. Show your work.

Tiles Used

	Blue	White	Green
Bath #1	20	50	10
Bath #2	15	30	5
Kitchen	25	100	50

[3] appropriate solution strategy with a minor computational or copying error
[2] correct equation solved incorrectly OR incorrect equation solved correctly; total cost correct based on previous work
[1] correct total cost with no work shown
[0] incorrect answers and no work shown OR no answers given

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12-2 Enrichment
Matrix Multiplication

Nilpotent Matrices

A matrix A is said to be nilpotent if there is an integer n such that $A^n = 0$, the zero matrix.

1. What can you say about the dimensions of a nilpotent matrix? Why?

If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, find A^2 and A^3. What can you conclude? The number of rows must be the same as the number of columns, so it can be multiplied by itself; $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; A nonzero matrix may be nilpotent.

The *order* of a nilpotent matrix A is the least integer n such that $A^n = 0$. In the example above, although both $A^2 = 0$ and $A^3 = 0$, the order of matrix A is 2.

2. Examine the conditions under which a nonzero 2×2 matrix of order 2 is nilpotent. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find A^2. $\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

If A is nilpotent of order 2, then $A^2 = 0$. Therefore, each element of the matrix A^2 must be equal to zero.

3. Write four equations that show that the elements in the corresponding rows and columns are zero. $a^2 + bc = 0$; $ab + bd = 0$; $ac + cd = 0$; $bc + d^2 = 0$

Equation (1, 1): Equation (1, 2):

Equation (2, 1): Equation (2, 2):

4. Factor equation (1, 2). $b(a + d) = 0$

5. Find two possible solutions. $b = 0$; $a + d = 0$

6. If $b = 0$, what can you conclude from equation (1, 1)? From equation (2, 2)? $a^2 = 0$, so $a = 0$; $d^2 = 0$, so $d = 0$

7. Use this information to write matrix A. Verify that this matrix is nilpotent of order 2. $A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$; $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8. Using equations (2, 1), (1, 1), and (2, 2), and substituting in matrix A, find another nilpotent matrix of order 2.

$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

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12-2 Reteaching
Matrix Multiplication

- To multiply a matrix by a real number, multiply each element in the matrix by the real number. This is *scalar multiplication*. The real number is the *scalar*.
- Solving matrix equations with scalars is like solving other kinds of equations. Isolate the variable on one side of the equal sign and simplify the other side.

Problem

What is the solution of $-3\begin{bmatrix} 1 & 3 \\ 6 & 4 \end{bmatrix} + 2X = \begin{bmatrix} -3 & 1 \\ -20 & -6 \end{bmatrix}$?

$\begin{bmatrix} -3 \cdot 1 & -3 \cdot 3 \\ -3 \cdot 6 & -3 \cdot 4 \end{bmatrix} + 2X = \begin{bmatrix} -3 & 1 \\ -20 & -6 \end{bmatrix}$ Multiply $\begin{bmatrix} 1 & 3 \\ 6 & 4 \end{bmatrix}$ by the scalar -3.

$\begin{bmatrix} -3 & -9 \\ -18 & -12 \end{bmatrix} + 2X = \begin{bmatrix} -3 & 1 \\ -20 & -6 \end{bmatrix}$ Simplify.

$2X = \begin{bmatrix} -3 & 1 \\ -20 & -6 \end{bmatrix} - \begin{bmatrix} -3 & -9 \\ -18 & -12 \end{bmatrix}$ Subtract $\begin{bmatrix} -3 & -9 \\ -18 & -12 \end{bmatrix}$ from each side.

$2X = \begin{bmatrix} 0 & 10 \\ -2 & 6 \end{bmatrix}$ Simplify.

$\frac{1}{2}(2X) = \frac{1}{2}\begin{bmatrix} 0 & 10 \\ -2 & 6 \end{bmatrix}$ Multiply each side by $\frac{1}{2}$ to isolate X.

$X = \begin{bmatrix} \frac{1}{2} \cdot 0 & \frac{1}{2} \cdot 10 \\ \frac{1}{2} \cdot (-2) & \frac{1}{2} \cdot 6 \end{bmatrix}$ Multiply $\begin{bmatrix} 0 & 10 \\ -2 & 6 \end{bmatrix}$ by the scalar $\frac{1}{2}$.

$X = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$ Simplify.

Exercises

Solve each matrix equation.

1. $\begin{bmatrix} 5 & -1 \\ 0 & 3 \end{bmatrix} + X = 2\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$ $\begin{bmatrix} -11 & 9 \\ 8 & 3 \end{bmatrix}$

2. $\frac{2}{3}\begin{bmatrix} 9 & 12 \\ 3 & -6 \end{bmatrix} = 4X + \begin{bmatrix} -6 & 0 \\ 0 & -3 \end{bmatrix}$ $\begin{bmatrix} 3 & 2 \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 3X = \begin{bmatrix} -2 & -7 & 6 \\ 3 & -3 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & -2 \\ -1 & 1 & -2 \end{bmatrix}$

4. $2\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \end{bmatrix} - 4X$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

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12-2 Reteaching (continued)
Matrix Multiplication

For two matrices A and B:

- Matrix multiplication $A \cdot B = C$ is defined when the number of columns in matrix A equals the number of rows in matrix B.
- The product matrix C has the same number of rows as matrix A and the same number of columns as matrix B.

Problem

If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ 2 & 5 \end{bmatrix}$, what is AB?

Step 1 Check if multiplication is defined for these matrices.
A has 3 columns and B has 3 rows, multiplication is defined.

Step 2 Multiply the first row of A by the first column of B. Add the products and write the sum in $(AB)_{11}$.

$(3)(1) + (1)(3) + (-1)(2) = 4$ $\begin{bmatrix} 4 & \square \\ \square & \square \end{bmatrix}$

Step 3 Multiply the first row of A by the second column of B. Add the products and write the sum in $(AB)_{12}$.

$(3)(4) + (1)(-1) + (-1)(5) = 6$ $\begin{bmatrix} 4 & 6 \\ \square & \square \end{bmatrix}$

Step 4 Multiply the second row of A by the first column of B. Add the products and write the sum in $(AB)_{21}$.

$(2)(1) + (0)(3) + (3)(2) = 8$ $\begin{bmatrix} 4 & 6 \\ 8 & \square \end{bmatrix}$

Step 5 Multiply the second row of A by the second column of B. Add the products and write the sum in $(AB)_{22}$.

$(2)(4) + (0)(-1) + (3)(5) = 23$ $\begin{bmatrix} 4 & 6 \\ 8 & 23 \end{bmatrix}$

Exercises

Find each product.

5. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 \\ -1 & 11 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 0 \\ -3 & 5 \end{bmatrix}$ $\begin{bmatrix} 0 & 26 \\ -8 & 3 \\ -13 & 33 \\ -7 & 32 \end{bmatrix}$

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12-3 ELL Support
Determinants and Inverses

Choose the word from the list that best matches each sentence.

determinant	multiplicative identity matrix	multiplicative inverse matrix
singular matrix	square matrix	

1. The ___determinant___ of $\begin{bmatrix}4 & 1\\3 & 2\end{bmatrix}$ is $8 - 3 = 5$.

2. A ___square matrix___ has the same number of rows and columns.

3. The ___multiplicative inverse matrix___ of matrix A can be written as A^{-1}.

4. A matrix with a determinant of zero is called a ___singular matrix___.

5. A matrix multiplied by its inverse produces the ___multiplicative identity matrix___.

Circle the determinant of the following matrices.

6. $\begin{bmatrix}6 & 5\\1 & 2\end{bmatrix}$ — **(A) 7** (B) 12 (C) 28

7. $\begin{bmatrix}3 & 6\\2 & 4\end{bmatrix}$ — (A) 18 (B) 10 **(C) 0**

8. $\begin{bmatrix}2 & 1 & -2\\-1 & -4 & 3\\4 & 6 & 5\end{bmatrix}$ — **(A) −79** (B) 47 (C) 79

Determine whether the following matrices are inverses.

9. $A = \begin{bmatrix}2 & -5\\-1 & 4\end{bmatrix}$ $B = \begin{bmatrix}\frac{4}{3} & \frac{5}{3}\\\frac{1}{3} & \frac{2}{3}\end{bmatrix}$ ___inverses___

10. $A = \begin{bmatrix}3 & 6\\1 & 4\end{bmatrix}$ $B = \begin{bmatrix}\frac{1}{2} & 1\\\frac{1}{6} & \frac{2}{3}\end{bmatrix}$ ___not inverses___

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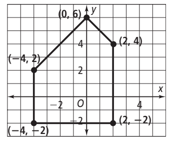
12-3 Think About a Plan
Determinants and Inverses

Geometry Find the area of the figure to the right.



Understanding the Problem

1. You know how to find the area of what shape using matrices? **triangle**

2. Can you divide the figure into these shapes? Explain.
Answers may vary. Sample: Yes; by picking a vertex and drawing two segments from it to the other two nonadjacent vertices, I can divide the figure into 3 triangles

3. What is the problem asking you to find?
the area of the figure by dividing it into triangles and adding the areas of the triangles

Planning the Solution

4. Divide the figure into these shapes. List the vertices of the shapes.
Answers may vary. Sample: (0, 6), (2, 4), (−4, 2); (2, 4), (2, −2), (−4, 2); (2, −2), (−4, −2), (−4, 2)

5. Write an expression to find the area of the figure. **Answers may vary. Sample:**
$$\frac{1}{2}\left|\det\begin{bmatrix}0 & 6 & 1\\2 & 4 & 1\\-4 & 2 & 1\end{bmatrix}\right| + \frac{1}{2}\left|\det\begin{bmatrix}2 & 4 & 1\\2 & -2 & 1\\-4 & 2 & 1\end{bmatrix}\right| + \frac{1}{2}\left|\det\begin{bmatrix}2 & -2 & 1\\-4 & -2 & 1\\-4 & 2 & 1\end{bmatrix}\right|$$

Getting an Answer

6. Simplify your expression to find the area of the figure. **38 units^2**

7. Is your answer reasonable? Explain.
Answers may vary. Sample: Yes; the figure fits within a rectangle that is 6 units wide and 8 units high, so an area somewhat less than 48 units^2 is reasonable

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12-3 Practice
Form G

Determinants and Inverses

Determine whether the matrices are multiplicative inverses.

1. $\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}, \begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}$ **yes**

2. $\begin{bmatrix}4 & 9\\2 & 6\end{bmatrix}, \begin{bmatrix}1 & -\frac{3}{2}\\-\frac{1}{3} & \frac{2}{3}\end{bmatrix}$ **yes**

3. $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \begin{bmatrix}-2 & 1\\\frac{3}{2} & -\frac{1}{2}\end{bmatrix}$ **yes**

4. $\begin{bmatrix}2 & 1 & 1\\1 & 1 & 2\\1 & 2 & 1\end{bmatrix}, \begin{bmatrix}\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2}\\-\frac{1}{2} & -\frac{1}{2} & \frac{3}{2}\\-\frac{1}{2} & \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$ **no**

5. $\begin{bmatrix}2 & 3 & 1\\-1 & 3 & -2\\1 & 2 & 0\end{bmatrix}, \begin{bmatrix}-\frac{4}{3} & -\frac{2}{3} & 3\\\frac{2}{3} & \frac{1}{3} & -1\\\frac{5}{3} & \frac{1}{3} & -3\end{bmatrix}$ **yes**

Evaluate the determinant of each matrix.

6. $\begin{bmatrix}-3 & 4\\1 & -1\end{bmatrix}$ **−1**

7. $\begin{bmatrix}3 & 9\\3 & 2\end{bmatrix}$ **−21**

8. $\begin{bmatrix}1 & -4\\2 & 6\end{bmatrix}$ **14**

9. $\begin{bmatrix}4 & -3\\1 & -8\end{bmatrix}$ **−29**

10. $\begin{bmatrix}5 & 4\\4 & 5\end{bmatrix}$ **9**

11. $\begin{bmatrix}1 & -12\\3 & 0\end{bmatrix}$ **36**

12. $\begin{bmatrix}1 & 2 & -2\\0 & 3 & 2\\1 & -1 & 3\end{bmatrix}$ **21**

13. $\begin{bmatrix}0 & 2 & 3\\4 & 1 & -2\\-2 & 3 & 1\end{bmatrix}$ **42**

14. $\begin{bmatrix}8 & -1 & 0\\0 & 0 & 2\\9 & 12 & -4\end{bmatrix}$ **−210**

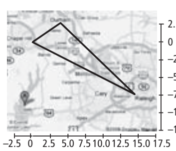
Graphing Calculator Evaluate the determinant of each 3×3 matrix.

15. $\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$ **−1**

16. $\begin{bmatrix}5 & 6 & 7\\-2 & 9 & 10\\8 & -1 & 4\end{bmatrix}$ **268**

17. $\begin{bmatrix}5.4 & 2.6 & 1.9\\-5.5 & 5.1 & 8.2\\4.8 & -8.2 & 2.7\end{bmatrix}$ **617.578**

18. The area between the North Carolina cities of Raleigh, Durham, and Chapel Hill is called the Research Triangle. Use the map to determine the approximate area of the Research Triangle. The coordinates are given in miles.
Answers may vary. Sample: about 32 mi^2



Source: Google Maps

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12-3 Practice (continued)
Form G

Determinants and Inverses

Determine whether each matrix has an inverse. If an inverse matrix exists, find it.

19. $\begin{bmatrix}3 & 4\\-3 & 4\end{bmatrix}$ $\begin{bmatrix}\frac{1}{6} & -\frac{1}{6}\\\frac{1}{8} & \frac{1}{8}\end{bmatrix}$

20. $\begin{bmatrix}3 & 4\\3 & 4\end{bmatrix}$ **no inverse**

21. $\begin{bmatrix}4 & 3\\3 & 2\end{bmatrix}$ $\begin{bmatrix}-2 & 3\\3 & -4\end{bmatrix}$

22. $\begin{bmatrix}30 & -4\\-25 & 3\end{bmatrix}$ $\begin{bmatrix}-\frac{3}{10} & -\frac{2}{5}\\-\frac{5}{2} & -3\end{bmatrix}$

23. $\begin{bmatrix}5 & 0\\-5 & 1\end{bmatrix}$ $\begin{bmatrix}\frac{1}{5} & 0\\1 & 1\end{bmatrix}$

24. $\begin{bmatrix}-12 & 4\\-9 & 3\end{bmatrix}$ **no inverse**

25. Use the coding matrix $\begin{bmatrix}3 & 6\\9 & 12\end{bmatrix}$ to encode the serial number 45-8-62-4-31-10.
159-210-246-453-444-678

Evaluate each determinant.

26. $\begin{bmatrix}7 & 3\\-6 & 4\end{bmatrix}$ **46**

27. $\begin{bmatrix}-5 & 3\\3 & 8\end{bmatrix}$ **−49**

28. $\begin{bmatrix}-2 & -2\\-2 & -4\end{bmatrix}$ **4**

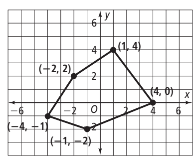
29. $\begin{bmatrix}3 & -2 & 5\\1 & 0 & -4\\-4 & 3 & 9\end{bmatrix}$ **37**

30. $\begin{bmatrix}4 & 4 & 4\\3 & 3 & 3\\1 & -1 & 3\end{bmatrix}$ **0**

31. $\begin{bmatrix}7 & 4 & -3\\6 & 10 & -1\\8 & 0 & 8\end{bmatrix}$ **576**

32. **Writing** Describe how to use matrices to find the area of a polygon.
Answers may vary. Sample: Divide the polygon into triangles. Name the coordinates of each vertex. Find the area of each triangle using the formula
$$Area = \frac{1}{2}\left|\det\begin{bmatrix}x_1 & y_1 & 1\\x_2 & y_2 & 1\\x_3 & y_3 & 1\end{bmatrix}\right|.$$ **Add the areas.**

33. Find the area of the figure at the right.
25.5 square units



Determine whether each matrix has an inverse. If an inverse matrix exists, find it. If it does not exist, explain why not.

34. $\begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}$ $\begin{bmatrix}1 & -0.75\\0 & 0.25\end{bmatrix}$

35. $\begin{bmatrix}0 & 2\\-1 & -1\end{bmatrix}$ $\begin{bmatrix}-0.5 & -1\\0.5 & 0\end{bmatrix}$

36. $\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 1\end{bmatrix}$ $\begin{bmatrix}-1 & 0 & 1\\0 & -1 & 1\\1 & 1 & -1\end{bmatrix}$

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12-3 Practice — Form K
Determinants and Inverses

Determine whether the following matrices are multiplicative inverses.

1. $\begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -0.5 \\ -3 & 1 \end{bmatrix}$ yes

2. $\begin{bmatrix} 2 & -4 & 1 \\ 6 & -3 & -7 \\ 9 & 5 & -2 \end{bmatrix}, \begin{bmatrix} 8 & 3 & -3 \\ -9 & 2 & 7 \\ 4 & -1 & -6 \end{bmatrix}$ no

3. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ yes

Evaluate the determinant of each matrix.

To start, write the formula for the determinant of a 2×2 matrix.

4. $\begin{bmatrix} 4 & 1 \\ -5 & 3 \end{bmatrix}$ 17

5. $\begin{bmatrix} -1 & 3 & -4 \\ 6 & -2 & 8 \\ 5 & 7 & -3 \end{bmatrix}$ 16

6. $\begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$ 2

7. $\begin{bmatrix} 2 & 0 & -4 \\ -1 & 3 & 2 \\ -2 & 1 & 4 \end{bmatrix}$ 0

8. $\begin{bmatrix} -2 & -3 \\ 5 & 7 \end{bmatrix}$ 1

9. $\begin{bmatrix} 3 & 4 & -1 \\ 1 & 8 & 3 \\ 5 & 2 & 2 \end{bmatrix}$ 120

10. **Error Analysis** Your friend evaluated the determinant of the matrix $\begin{bmatrix} -6 & -7 \\ 3 & 2 \end{bmatrix}$ and got -9. What error did your friend make, and what is the correct determinant?
Your friend subtracted the product of *ad* from the product of *bc* rather than subtracting *bc* from *ad*. The correct determinant is 9.

11. **Open-Ended** Write a 2×2 matrix with a determinant of zero.
Answers may vary. Sample: $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$

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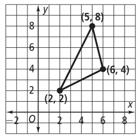
12-3 Practice (continued) — Form K
Determinants and Inverses

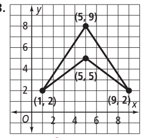
Use matrices to find the areas of the following triangles. Express your answers in square units.

12. 

$\text{Area} = \frac{1}{2}\left|\det A\right|$

$\text{Area} = \frac{1}{2}\det\begin{bmatrix} 2 & 2 & 1 \\ 6 & 4 & 1 \\ 5 & 8 & 1 \end{bmatrix}$

$\text{Area} = \boxed{9 \text{ units}^2}$

13. 

16 units^2

Find the inverse of each matrix, if one exists.

To start, find the determinant of the matrix.

14. $A = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}$

$\det A = 6(1) - 2(2) = 2$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & 3 \end{bmatrix}$

15. $A = \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$

16. $A = \begin{bmatrix} 10 & 5 \\ 4 & 2 \end{bmatrix}$ *A* has no inverse.

17. Your aunt's checking account number is 6143-0571-2943-3072. Use the coding matrix $C = \begin{bmatrix} -2 & 1 \\ -1 & 3 \end{bmatrix}$ to encode the account number.

$\begin{bmatrix} -10 & 7 & -4 & -3 & 3 & -10 & -7 & 0 \\ 0 & 26 & 8 & 6 & 9 & -5 & 14 & 5 \end{bmatrix}$

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12-3 Standardized Test Prep
Determinants and Inverses

Gridded Response

Solve each exercise and enter your answer in the grid provided.

1. What is the determinant of $\begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix}$?

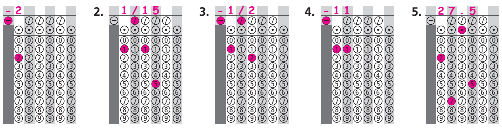
2. If $A = \begin{bmatrix} 2 & 1 \\ -9 & 3 \end{bmatrix}$ and the inverse of A is $x \cdot \begin{bmatrix} 3 & -1 \\ 9 & 2 \end{bmatrix}$ what is the value of x?

3. If $\begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} x & 1 \\ 2 & -3 \end{bmatrix}$ what is the value of x?

4. What is the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 0 & 3 & 2 \end{bmatrix}$?

5. What is the area of a triangle with vertices at $(-5, 0)$, $(3, -1)$, and $(2, 6)$?

Answers



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12-3 Enrichment
Determinants and Inverses

Suppose A and B are 2×2 matrices as follows:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

1. What is the value of the determinant of A, $\det A$? $ad - bc$

2. Evaluate $\det B$. $eh - fg$

3. Evaluate $\det A \cdot \det B$. $adeh + bcfg - bceh - adfg$

4. Compute matrix AB. $\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

5. Evaluate $\det(AB)$. $(ae + bg)(cf + dh) - (af + bh)(ce + dg) = adeh + bcfg - bceh - adfg$

6. What can you conclude? $\det(AB) = \det A \cdot \det B$

7. Explain your results.
The determinant of the product of two matrices is equal to the product of the determinants.

8. Suppose that a 2×2 matrix A has an inverse A^{-1}. Use the product rule to investigate how the determinant of A^{-1} is related to the determinant of A. $\det I$; 1; $\frac{1}{\det A}$

$\det(A \cdot A^{-1}) = \det A \cdot \det A^{-1}$

__________ $= \det A \cdot \det A^{-1}$

__________ $= \det A \cdot \det A^{-1}$

__________ $= \det A^{-1}$

9. Explain your results.
The determinant of the inverse of a matrix is equal to the reciprocal (inverse) of the determinant of the original matrix.

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12-3 Reteaching

Determinants and Inverses

The determinant of a matrix combines elements according to a pattern.

2 × 2 matrix $X = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\det X = a_1b_2 - a_2b_1$ Notice the patterns in the formula for the determinant. Variables: $ab - ab$ Subscripts: $12 - 21$
3 × 3 matrix $Y = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$	$\det Y = (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$ Notice the patterns in the formula for the determinant. Variables: $(abc + bca + cab) - (abc + bca + cab)$ Subscripts: $(123 + 123 + 123) - (321 + 321 + 321)$

Problem

Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. What is the determinant of B?

Step 1 Name the matrix elements.

$a_1 = 1, b_1 = 2, c_1 = 3, a_2 = 4, b_2 = 5, c_2 = 6, a_3 = 7, b_3 = 8, c_3 = 9$

Step 2 Substitute in the determinant formula and simplify.

$$\begin{aligned}\det B &= (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1) \\ &= (1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8) - (7 \cdot 5 \cdot 3 + 8 \cdot 6 \cdot 1 + 9 \cdot 4 \cdot 2) \\ &= (45 + 84 + 96) - (105 + 48 + 72) \\ &= 0\end{aligned}$$

Exercises

Evaluate the determinant of each matrix.

1. $\begin{bmatrix} 3 & 3 \\ 8 & 1 \end{bmatrix}$ **−21**

2. $\begin{bmatrix} -3 & -2 \\ -5 & -1 \end{bmatrix}$ **−7**

3. $\begin{bmatrix} 4 & -7 \\ 2 & 9 \end{bmatrix}$ **50**

4. $\begin{bmatrix} 6 & 2 & -3 \\ -1 & 8 & 0 \\ 3 & -2 & 4 \end{bmatrix}$ **266**

5. $\begin{bmatrix} 7 & 0 & 3 \\ 0 & 8 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ **28**

6. $\begin{bmatrix} -3 & 3 & 9 \\ 1 & -2 & 5 \\ 5 & 1 & -6 \end{bmatrix}$ **171**

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12-3 Reteaching (continued)

Determinants and Inverses

- The inverse of matrix A is written A^{-1}.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, for $ad - bc \neq 0$.
- $AA^{-1} = I$, where I is the identity matrix.

Problem

Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. What is A^{-1}?

$ad - bc = (2)(3) - (4)(1) = 2$ — Find $ad - bc$. Check that $ad - bc \neq 0$.

$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$ — Substitute values in the matrix inverse formula.

$A^{-1} = \begin{bmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$ — Multiply each element by $\frac{1}{\det A}$ to simplify.

Check your work. The product of a matrix and its inverse should equal the identity matrix.

$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ — Substitute in the formula $AA^{-1} = I$.

$\begin{bmatrix} 2\left(\frac{3}{2}\right) + 4\left(-\frac{1}{2}\right) & 2(-2) + 4(1) \\ 1\left(\frac{3}{2}\right) + 3\left(-\frac{1}{2}\right) & 1(-2) + 3(1) \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ — Multiply.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✔ — Simplify.

Exercises

Find the inverse of each matrix, if possible. If no inverse exists, write *no inverse*.

7. $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 9 & -2 \\ 5 & -1 \end{bmatrix}$ $\begin{bmatrix} -1 & 2 \\ -5 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$ **no inverse**

11. $\begin{bmatrix} -2 & 17 \\ 1 & 8 \end{bmatrix}$ $\begin{bmatrix} -\frac{8}{33} & \frac{17}{33} \\ \frac{1}{33} & \frac{2}{33} \end{bmatrix}$

12. $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$

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12-4 ELL Support

Inverse Matrices and Systems

Problem

What is the solution of the matrix equation $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}X = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$? Explain your steps.

$\det A = (3)(2) - (5)(1) = 1$ — Find the determinant of the coefficient matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

$A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ — Use the determinant to find A^{-1}.

$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}X = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ — Multiply both sides of the original equation on the left by A^{-1}.

$\begin{bmatrix} (2)(3) + (-1)(5) & (2)(1) + (-1)(2) \\ (-5)(3) + (3)(5) & (-5)(1) + (3)(2) \end{bmatrix}X = \begin{bmatrix} (2)(2) + (-1)(-4) \\ (-5)(2) + (3)(-4) \end{bmatrix}$ — Perform the matrix multiplication.

$X = \begin{bmatrix} 8 \\ -22 \end{bmatrix}$ — Simplify.

Exercise

What is the solution of the matrix equation $\begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$? Explain your steps.

$\det A = (5)(2) - (4)(2) = 2$ — **Find the determinant of the coefficient matrix $A = \begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}$.**

$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & \frac{5}{2} \end{bmatrix}$ — **Use the determinant to find A^{-1}.**

$\begin{bmatrix} 1 & -2 \\ -1 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}X = \begin{bmatrix} 1 & -2 \\ -1 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ — **Multiply both sides of the original equation on the left by A^{-1}.**

$\begin{bmatrix} (1)(5) + (-2)(2) & (1)(4) + (-2)(2) \\ (-1)(5) + \left(\frac{5}{2}\right)(2) & (-1)(4) + \left(\frac{5}{2}\right)(2) \end{bmatrix}X = \begin{bmatrix} (1)(3) + (-2)(-2) \\ (-1)(3) + \left(\frac{5}{2}\right)(-2) \end{bmatrix}$ — **Perform the matrix multiplication.**

$X = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$ — **Simplify.**

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12-4 Think About a Plan

Inverse Matrices and Systems

Nutrition Suppose you are making a trail mix for your friends and want to fill three 1-lb bags. Almonds cost $2.25/lb, peanuts cost $1.30/lb, and raisins cost $.90/lb. You want each bag to contain twice as much nuts as raisins by weight. If you spent $4.45, how much of each ingredient did you buy?

Know

1. I need **3 lb** of ingredients that cost a total of **$4.45**.
2. **Almonds cost $2.25/lb, peanuts cost $1.30/lb, and raisins cost $.90/lb**.
3. **Each bag will contain twice as much nuts as raisins by weight**.

Need

4. To solve the problem I need to: **write and solve 3 equations for three unknowns**.

Plan

5. Let x = the number of pounds of almonds, y = the number of pounds peanuts, and z = the number of pounds of raisins. Write a system of equations that solve the problem.

$x + y + z = 3$
$x + y - 2z = 0$
$2.25x + 1.3y + 0.9z = 4.45$

6. Write the system as a matrix equation. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 2.25 & 1.3 & 0.9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4.45 \end{bmatrix}$

7. Use a calculator. Solve for the variable matrix. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

8. How much of each ingredient did you buy?
1 lb of almonds, 1 lb of peanuts, and 1 lb of raisins

9. How can you check your solution? Does your solution check?
Substitute $x = 1$, $y = 1$, and $z = 1$ into the system of equations. The solution checks.

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12-4 Practice — Form G
Inverse Matrices and Systems

Solve each matrix equation. If an equation cannot be solved, explain why.

1. $\begin{bmatrix} 0.25 & -0.75 \\ 3.5 & 2.25 \end{bmatrix} X = \begin{bmatrix} 1.5 \\ -3.75 \end{bmatrix}$ $\begin{bmatrix} \frac{3}{17} \\ -\frac{33}{17} \end{bmatrix}$

2. $\begin{bmatrix} 3 & -9 \\ 1 & -6 \end{bmatrix} X = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 8 \\ \frac{4}{3} \end{bmatrix}$

3. $\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} X = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$ no sol.; det $A = 0$

4. $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} X = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$ $\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{bmatrix}$

Write each system as a matrix equation. Identify the coefficient matrix, the variable matrix, and the constant matrix.

5. $\begin{cases} 6x + 9y = 36 \\ 4x + 13y = 2 \end{cases}$
$\begin{bmatrix} 6 & 9 \\ 4 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \end{bmatrix}$; $\begin{bmatrix} 6 & 9 \\ 4 & 13 \end{bmatrix}$; $\begin{bmatrix} x \\ y \end{bmatrix}$; $\begin{bmatrix} 36 \\ 2 \end{bmatrix}$

6. $\begin{cases} 3x - 4y = -9 \\ 7y = 24 \end{cases}$
6. $\begin{bmatrix} 3 & -4 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ 24 \end{bmatrix}$; $\begin{bmatrix} 3 & -4 \\ 0 & 7 \end{bmatrix}$; $\begin{bmatrix} x \\ y \end{bmatrix}$; $\begin{bmatrix} -9 \\ 24 \end{bmatrix}$

7. $\begin{cases} 3a = 5 \\ b = 12 + a \end{cases}$
$\begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$; $\begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}$; $\begin{bmatrix} a \\ b \end{bmatrix}$; $\begin{bmatrix} 5 \\ 12 \end{bmatrix}$

8. $\begin{cases} 4x - z = 9 \\ 12x + 2y = 17 \\ x - y + 12z = 3 \end{cases}$
8. $\begin{bmatrix} 4 & 0 & -1 \\ 12 & 2 & 0 \\ 1 & -1 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ 3 \end{bmatrix}$; $\begin{bmatrix} 4 & 0 & -1 \\ 12 & 2 & 0 \\ 1 & -1 & 12 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $\begin{bmatrix} 9 \\ 17 \\ 3 \end{bmatrix}$

Solve each system of equations using a matrix equation. Check your answers.

9. $\begin{cases} x + 3y = 5 \\ x + 4y = 6 \end{cases}$ $x = 2, y = 1$

10. $\begin{cases} 2x + 3y = 12 \\ x + 2y = 7 \end{cases}$ $x = 3,$ $y = 2$

11. $\begin{cases} x - 3y = -1 \\ -6x + 19y = 6 \end{cases}$ $x = -1,$ $y = 0$

12. $\begin{cases} 4x - 3y = 55 \\ x + y = 5 \end{cases}$ $x = 10,$ $y = -5$

13. $\begin{cases} 6x + 7y = -12 \\ 3x - 4y = -6 \end{cases}$ $x = -2,$ $y = 0$

14. $\begin{cases} 3x - y = 6 \\ -2x + 3y = 10 \end{cases}$ $x = 4,$ $y = 6$

15. $\begin{cases} -3x + 4y - z = -5 \\ x - y - z = -8 \\ 2x + y + 2z = 9 \end{cases}$ $x = -2,$ $y = -1,$ $z = 7$

16. $\begin{cases} x + y + z = 31 \\ x - y + z = 1 \\ x - 2y + 2z = 7 \end{cases}$ $x = -5,$ $y = 15,$ $z = 21$

17. $\begin{cases} x + 2y - z = 8 \\ -2x + 3z = -4 \\ y + z = 3 \end{cases}$ $x = 2,$ $y = 3,$ $z = 0$

18. $\begin{cases} 3x - 2y + 4z = -10 \\ y - 3z = 1 \\ 2x + z = -3 \end{cases}$ $x = -2,$ $y = 4,$ $z = 1$

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12-4 Practice (continued) — Form G
Inverse Matrices and Systems

19. An apartment building has 50 units. All have one or two bedrooms. One-bedroom units rent for \$425/mo. Two-bedroom units rent for \$550/mo. When all units are occupied, the total monthly rent collected is \$25,000. How many units of each type are in the building? **20 one-bedroom, 30 two-bedroom**

20. The difference between twice Bill's age and Carlos's age is 26. The sum of Anna's age, three times Bill's age, and Carlos's age is 92. The total of the three ages is 52.
  a. Write a matrix equation to represent this situation. $\begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 26 \\ 92 \\ 52 \end{bmatrix}$
  b. How old is each person? **Anna: 18, Bill: 20, Carlos: 14**

Solve each system.

21. $\begin{cases} x + 2y - 3z = 18 \\ -3x - z = -20 \\ y + 3z = -13 \end{cases}$ $x = 8, y = -1,$ $z = -4$

22. $\begin{cases} x + y + 3z = 9 \\ 2y - 5z = -21 \\ 2x - 5y = 21 \end{cases}$ $x = 3, y = -3,$ $z = 3$

23. $\begin{cases} w + 2x - 3y + z = -2 \\ 2w - x - y + 3z = 3 \\ -w + 3x + y - z = 0 \\ 3w - x - 2y + 2z = -1 \end{cases}$ $w = -1,$ $x = 0,$ $y = 1,$ $z = 2$

24. $\begin{cases} 2w + 3x - y + z = -11 \\ w + x + y + z = 0 \\ -3w - 2x - y - z = -3 \\ -2w + x + 3y + 2z = -5 \end{cases}$ $w = 3,$ $x = -3,$ $y = 4,$ $z = -4$

Solve each matrix equation. If the coefficient matrix has no inverse, write *no unique solution*.

25. $\begin{bmatrix} 12 & -3 \\ 16 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 144 \\ -64 \end{bmatrix}$ $x = 4, y = -32$

26. $\begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$ no unique solution

Determine whether each system has a unique solution.

27. $\begin{cases} 4d + 2e = 4 \\ d + 3e = 6 \end{cases}$ yes

28. $\begin{cases} 3x - 2y = 43 \\ 9x - 6y = 40 \end{cases}$ no

29. $\begin{cases} -y - z = 3 \\ x + 2y + 3z = 1 \\ 4x - 5y - 6z = -50 \end{cases}$ yes

30. **Reasoning** Explain how you could use a matrix equation to show that the lines represented by $y = -3x + 4$ and $y = -4x - 8$ intersect. **Write the matrix equation $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$ to solve the system. If the equation has a solution (x, y), then the lines intersect at that point. A solution exists if the determinant of the coefficient matrix $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ does not equal zero.**

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12-4 Practice — Form K
Inverse Matrices and Systems

Solve each matrix equation.

To start, find the determinant of the coefficient matrix.

1. $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} X = \begin{bmatrix} 11 & 24 \\ 5 & 10 \end{bmatrix}$

$\det\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 5(1) - 2(2) = 1$

$A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$X = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 11 & 24 \\ 5 & 10 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}}$

2. $\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix} X = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$
$X = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Write each system as a matrix equation. Identify the coefficient matrix, the variable matrix, and the constant matrix.

3. $\begin{cases} 3x + y = 9 \\ 2x - 4y = -8 \end{cases}$
$\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \end{bmatrix}$

4. $\begin{cases} a + 4b = 13 \\ 3a + 2b = 19 \end{cases}$
$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \end{bmatrix}$

5. $\begin{cases} 5x + 3y = 35 \\ 2x = 38 - 6y \end{cases}$
$\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 35 \\ 38 \end{bmatrix}$

6. $\begin{cases} 3x + y = 9 \\ x - y + 4z = -5 \\ 3y + 2z = 7 \end{cases}$
$\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 4 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 7 \end{bmatrix}$

7. $\begin{cases} 4x = 2 - 2y \\ 3y = -12 - x \end{cases}$
$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$

8. $\begin{cases} 2a - 2c = -6 - b \\ 4a = 10 + c \\ 3c = 8 - 5b \end{cases}$
$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 0 & -1 \\ 0 & 5 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \\ 8 \end{bmatrix}$

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12-4 Practice (continued) — Form K
Inverse Matrices and Systems

Solve each system of two equations using a matrix equation.

9. $\begin{cases} 3x - y = 16 \\ 5x - 9y = 12 \end{cases}$

$\begin{bmatrix} 3 & -1 \\ 5 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$

$A^{-1} = -\frac{1}{22}\begin{bmatrix} -9 & 1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{9}{22} & -\frac{1}{22} \\ \frac{5}{22} & -\frac{3}{22} \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{9}{22} & -\frac{1}{22} \\ \frac{5}{22} & -\frac{3}{22} \end{bmatrix}\begin{bmatrix} 16 \\ 12 \end{bmatrix} = \boxed{\begin{bmatrix} 6 \\ 2 \end{bmatrix}}$

10. $\begin{cases} y = 32 - 4x \\ -2x = -2 - 3y \end{cases}$
$x = 7; y = 4$

11. $\begin{cases} 3a + 4b = -3 \\ 2a + 3b = -1 \end{cases}$
$a = -5; b = 3$

12. $\begin{cases} 2b = 3a - 14 \\ 2b = -20 + 4a \end{cases}$
$a = 6; b = 2$

Solve each system of three equations using a matrix equation.

13. $\begin{cases} 3x + 5y = 19 \\ x + 3y - 4z = 1 \\ 6x + 8z = 12 \end{cases}$
$x = -2; y = 5; z = 3$

14. $\begin{cases} 3b + 5c = 7 \\ 9 = a - 2c \\ 4a + 7c = 17 + b \end{cases}$
$a = 7; b = 4; c = -1$

15. $\begin{cases} 6y = 8 - x \\ x + y + z = -8 \\ 7y - 3z = 32 \end{cases}$
$x = -4; y = 2; z = -6$

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12-4 Standardized Test Prep

Inverse Matrices and Systems

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. Which matrix equation represents the system $\begin{cases} 2x - y = 11 \\ x + 3y = 2 \end{cases}$? C

Ⓐ $\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 2 & -1 & 11 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Ⓓ $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} 11 \\ 2 \end{bmatrix}$

2. Let $\begin{bmatrix} 3 & 5 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$. What values of x and y make the equation true? I

Ⓕ $(-12, -1)$ Ⓖ $(-4, -6)$ Ⓗ $(-3, -20)$ Ⓘ $(2, -2)$

3. Which system has a unique solution? C

Ⓐ $\begin{cases} 3x - 2y = 43 \\ 9x - 6y = 40 \end{cases}$

Ⓒ $\begin{cases} 2x - 5y = 6 \\ 4x + 7y = 12 \end{cases}$

Ⓑ $\begin{cases} 6x + 8y = 16 \\ -3x - 4y = 12 \end{cases}$

Ⓓ $\begin{cases} 4x + 2y = 10 \\ 8x + 4y = 18 \end{cases}$

4. Let $\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}X = \begin{bmatrix} 0 \\ -14 \end{bmatrix}$. What value of X makes the equation true? F

Ⓕ $\begin{bmatrix} -2 \\ 10 \end{bmatrix}$ Ⓖ $\begin{bmatrix} -6 \\ -15 \end{bmatrix}$ Ⓗ $\begin{bmatrix} 0 \\ 14 \end{bmatrix}$ Ⓘ $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

Short Response

5. The Spirit Club sold buttons for \$1, hats for \$4, and t-shirts for \$8. They sold 3 times as many buttons as hats. Together, the number of hats and t-shirts sold was equal to the number of buttons sold. They earned a total of \$460. Write and solve a matrix equation to find how many buttons, hats, and t-shirts the club sold.

[2] $\begin{bmatrix} 1 & 4 & 8 \\ 1 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} b \\ h \\ t \end{bmatrix} = \begin{bmatrix} 460 \\ 0 \\ 0 \end{bmatrix}$; $\begin{bmatrix} b \\ h \\ t \end{bmatrix} = \begin{bmatrix} 1 & 4 & 8 \\ 1 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 460 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 20 \\ 40 \end{bmatrix}$

They sold 60 buttons, 20 hats, and 40 t-shirts.

[1] incorrect or incomplete work shown

[0] incorrect answers and no work shown OR no answers given

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12-4 Enrichment

Inverse Matrices and Systems

You can find the determinant of a 3×3 matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ by using submatrices that you form by removing rows and columns.

You can form the submatrix M_{ij} by removing the ith row and the jth column. For example, M_{12} is the submatrix formed by removing the first row and the second column.

$$\begin{bmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ a_2 & \cancel{b_2} & c_2 \\ a_3 & \cancel{b_3} & c_3 \end{bmatrix}$$

So $M_{12} = \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix}$.

The determinant of a 3×3 matrix A is $\det A = a_1 \det M_{11} - b_1 \det M_{12} + c_1 \det M_{13}$.

1. Find the determinant of $C = \begin{bmatrix} -1 & 3 & 5 \\ 2 & -4 & 6 \\ 0 & 1 & -1 \end{bmatrix}$ using the formula above. 18

The determinant of a 4×4 matrix A can be found in a similar manner:

$\det A = a_1 \det M_{11} - b_1 \det M_{12} + c_1 \det M_{13} - d_1 \det M_{14}$

2. Find the determinant of $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 4 & 0 & 1 & 2 \\ 3 & 4 & 0 & 1 \end{bmatrix}$ using the formula above. 25

3. The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ has two identical rows. Calculate $\det A$. 0

4. Write a 3×3 matrix with two identical rows. Calculate its determinant.
Answers may vary. Sample: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$; 0

5. Write a 2×2 matrix with two identical rows. Calculate its determinant.
Answers may vary. Sample: $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$; 0

6. What do you think the determinant of an $n \times n$ matrix with two identical rows will be? Its determinant is zero.

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12-4 Reteaching

Inverse Matrices and Systems

- You can write the system $\begin{cases} ax + by = p \\ cx + dy = q \end{cases}$ as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$.
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the *coefficient matrix*, $\begin{bmatrix} x \\ y \end{bmatrix}$ is the *variable matrix*, and $\begin{bmatrix} p \\ q \end{bmatrix}$ is the *constant matrix*.
- Solve the matrix equation by multiplying both sides by the inverse of the coefficient matrix, if it exists.

Problem

What is the solution of the system $\begin{cases} 4x + 3y = -4 \\ 3x - y = -3 \end{cases}$? Solve using matrices.

$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$ Write the system as a matrix equation.

$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix} = \frac{1}{(-1)(4) - (3)(3)}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{13} & \frac{3}{13} \\ \frac{3}{13} & -\frac{4}{13} \end{bmatrix}$ Find the inverse of the coefficient matrix.

$\begin{bmatrix} \frac{1}{13} & \frac{3}{13} \\ \frac{3}{13} & -\frac{4}{13} \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{13} & \frac{3}{13} \\ \frac{3}{13} & -\frac{4}{13} \end{bmatrix}\begin{bmatrix} -4 \\ -3 \end{bmatrix}$ Isolate the variable matrix by multiplying both sides by the inverse of the coefficient matrix.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \left(-\frac{4}{13}\right) + \left(-\frac{9}{13}\right) \\ \left(-\frac{12}{13}\right) + \left(\frac{12}{13}\right) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ Simplify.

The solution is $(-1, 0)$. Substitute the values in the original system to check your work.

Exercises

Solve each system of equations using a matrix equation. Check your answers.

1. $\begin{cases} 2x - 7y = -3 \\ x + 5y = 7 \end{cases}$ $x = 2$, $y = 1$

2. $\begin{cases} x + 3y = 5 \\ x + 4y = 6 \end{cases}$ $x = 2$, $y = 1$

3. $\begin{cases} x - 3y = -1 \\ -5x + 16y = 5 \end{cases}$ $x = -1$, $y = 0$

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12-4 Reteaching (continued)

Inverse Matrices and Systems

You can use a graphing calculator and matrices to solve a linear system.

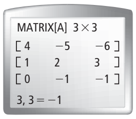
Problem

What is the solution of the system $\begin{cases} 4x - 5y - 6z = -50 \\ x + 2y + 3z = 1 \\ -y - z = 3 \end{cases}$? Solve using a graphing calculator and matrices.

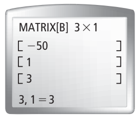
Write the system as a matrix equation.

$$\begin{bmatrix} 4 & -5 & -6 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -50 \\ 1 \\ 3 \end{bmatrix}$$

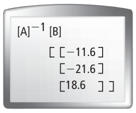
Enter the coefficient matrix as matrix A.



Enter the constant matrix as matrix B.



Multiply $A^{-1}B$ to find the variable matrix.



The solution is $(-11.6, -21.6, 18.6)$. Substitute the values in the original system to check your work.

Exercises

Solve each system of equations using a graphing calculator and matrices. Check your answers.

4. $\begin{cases} 4x + y + z = 0 \\ 5x + 2y + 3z = -15 \\ 6x - 5y - 5z = 52 \end{cases}$ $x = 2$, $y = 1$, $z = -9$

5. $\begin{cases} 0.5x + 1.5y + z = 7 \\ 3x + 3y + 5z = 3 \\ 2x + y + 2z = -1 \end{cases}$ $x = -1$, $y = 7$, $z = -9$

6. $\begin{cases} x + 2y - z = 19 \\ 2x - 3y = -8 \\ y - 5z = 16 \end{cases}$ $x = 5$, $y = 6$, $z = -2$

7. $\begin{cases} 4x + 8y + 3z = -7 \\ -y + 5z = -6 \\ 2x + 5y = -1 \end{cases}$ $x = -3$, $y = 1$, $z = -1$

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12-5 ELL Support
Geometric Transformations

For Exercises 1–5, draw a line from each word in Column A to its definition in Column B.

Column A	Column B
1. image	**A.** a transformation that enlarges or reduces an image
2. preimage	**B.** the point around which an image is rotated
3. dilation	**C.** a transformed figure
4. rotation	**D.** the original figure
5. center of rotation	**E.** a transformation that turns an image around a fixed point

For Exercises 6–9, draw a line from each matrix in Column A to the rotation that it represents in Column B.

Column A	Column B
6. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	**A.** 90° rotation
7. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	**B.** 180° rotation
8. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	**C.** 270° rotation
9. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	**D.** 360° rotation

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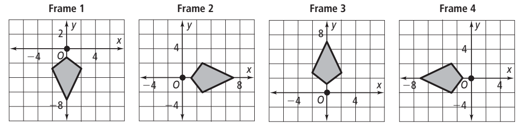
12-5 Think About a Plan
Geometric Transformations

Animation In an upcoming cartoon, the hero is a gymnast. In one scene he swings around a high bar, making two complete revolutions around the bar. What rotation matrices are needed so eight frames of the movie would show the illustrated motion, one frame after the other?



Understanding the Problem

1. Describe the rotation that occurs from one frame to the next. 90° counterclockwise

2. What is the problem asking you to determine?

   Answers may vary. Sample: The problem is asking me to determine the rotation matrices that are needed to show two complete revolutions around the high bar in eight frames.

Planning the Solution

3. Describe how the first four frames relate to the next four frames in the scene.

   Answers may vary. Sample: Frame 5 = Frame 1, Frame 6 = Frame 2, Frame 7 = Frame 3, Frame 8 = Frame 4

4. How many rotation matrices do you need? Explain.

   Answers may vary. Sample: 1; The next frame in the sequence is the product of the 90° counterclockwise rotation matrix and the matrix that represents the current frame. Use this process 7 times.

Getting an Answer

5. What rotation matrices are needed so eight frames of the movie would show the illustrated motion, one frame after the other? $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

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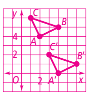
12-5 Practice
Form G

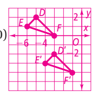
Geometric Transformations

Use matrix addition to find the coordinates of each image after a translation 2 units right and 4 units down. If possible, graph each pair of figures on the same coordinate plane.

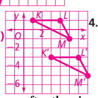
1. $A(2, 4)$, $B(4, 5)$, $C(1, 6)$
   $A'(4, 0)$, $B'(6, 1)$, $C'(3, 2)$



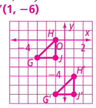
2. $D(-5, 2)$, $E(-6, 1)$, $F(-3, 0)$
   $D'(-3, -2)$, $E'(-4, -3)$, $F'(-1, -4)$



3. $K(1, 1)$, $L(4, 1)$, $M(5, -1)$
   $K'(3, -3)$, $L'(6, -3)$, $M'(7, -5)$



4. $G(-3, -2)$, $H(-1, 0)$, $J(-1, -2)$
   $G'(-1, -6)$, $H'(1, -4)$, $J'(1, -6)$



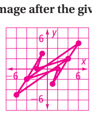
Find the coordinates of each image after the given dilation.

5. $\begin{bmatrix} 2 & -3 & 6 & 4 \\ 0 & 1 & 1 & -4 \end{bmatrix}$, 2
   (4, 0), (−6, 2), (12, 2), (8, −8)

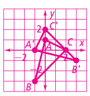
6. $\begin{bmatrix} -3 & 4 & 4 \\ 0 & 2 & -2 \end{bmatrix}$, 1.1
   (−3.3, 0), (4.4, 2.2), (4.4, −2.2)

Graph each figure and its image after the given rotation.

7. $\begin{bmatrix} 2 & 1 & 6 & -4 \\ 0 & -3 & 5 & -2 \end{bmatrix}$; 180°



8. $\begin{bmatrix} 0 & -1 & 2 \\ 1 & -3 & 0 \end{bmatrix}$, 90°



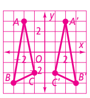
Find the coordinates of each image after the given rotation.

9. $\begin{bmatrix} -3 & -1 & 0 & 2 & 4 \\ 3 & 2 & 1 & 0 & -4 \end{bmatrix}$; 270°
   (3, 3), (2, 1), (1, 0), (0, −2), (−4, −4)

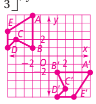
10. $\begin{bmatrix} 5 & 2 & 9 & 8 & 6 \\ 1 & -2 & 3 & 5 & -4 \end{bmatrix}$; 360°
    (5, 1), (2, −2), (9, 3), (8, 5), (6, −4)

Graph each figure and its image after reflection in the given line.

11. $\begin{bmatrix} -2 & -3 & -1 \\ 3 & -3 & -2 \end{bmatrix}$; y-axis



12. $\begin{bmatrix} -2 & -2 & -4 & -5 & -5 \\ 5 & 1 & 2 & 1 & 3 \end{bmatrix}$; $y = x$



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12-5 Practice (continued)
Form G

Geometric Transformations

Find the coordinates of each image after reflection in the given line.

13. $\begin{bmatrix} 9 & 3 & -2 & 4 \\ 5 & 1 & 0 & 6 \end{bmatrix}$; $y = -x$
    (−5, −9), (−1, −3), (0, 2), (−6, −4)

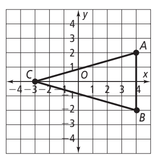
14. $\begin{bmatrix} 2 & -3 & -2 & 6 & 9 \\ 2 & 4 & -2 & -1 & 1 \end{bmatrix}$; x-axis
    (2, −2), (−3, −4), (−2, 2), (6, 1), (9, −1)

Geometry Each matrix represents the vertices of a polygon. Translate each figure 6 units right and 2 units up. Express your answer as a matrix.

15. $\begin{bmatrix} 2 & 3 & 3 & 2 & 1 & 1 \\ -2 & -3 & -4 & -5 & -4 & -3 \end{bmatrix}$
    $\begin{bmatrix} 8 & 9 & 9 & 8 & 7 & 7 \\ 0 & -1 & -2 & -3 & -2 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 7 \end{bmatrix}$
    $\begin{bmatrix} 8 & 9 & 7 \\ 5 & 7 & 9 \end{bmatrix}$

For Exercises 17–20, use $\triangle ABC$. Write the coordinates of each image in matrix form.



17. a dilation of 1.5 $\begin{bmatrix} 6 & 6 & -4.5 \\ 3 & -3 & 0 \end{bmatrix}$

18. a reflection across $\begin{bmatrix} 2 & -2 & 0 \\ 4 & 4 & -3 \end{bmatrix}$

19. a rotation of 270° $\begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & 3 \end{bmatrix}$

20. a translation 2 units right and 6 units down $\begin{bmatrix} 6 & 6 & -1 \\ -4 & -8 & -6 \end{bmatrix}$

Each pair of matrices represents the coordinates of the vertices of the preimage and image of a polygon. Describe the transformation.

21. $\begin{bmatrix} -1 & 2 & 4 & 5 \\ -4 & 4 & 5 & 2 \end{bmatrix}$, $\begin{bmatrix} -4 & -1 & 1 & 2 \\ -5 & 3 & 4 & 1 \end{bmatrix}$
    translation 3 units left and 1 unit down

22. $\begin{bmatrix} 4 & 12 & 20 & 16 \\ 8 & 4 & 0 & 12 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 5 & 4 \\ 2 & 1 & 0 & 3 \end{bmatrix}$
    a dilation of 0.25

23. **Writing** The matrices $\begin{bmatrix} -4 & -1 & -3 \\ 4 & 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 4 & 5 & 2 \\ -4 & -1 & -3 \end{bmatrix}$ represent the coordinates of the vertices of a triangle before and after a reflection in the line $y = x$. Describe the relationship between the coordinates of the corresponding vertices.
    Answers may vary. Sample: The values of the x-and y-coordinates have been exchanged.

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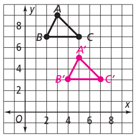
12-5 Practice *Form K*

Geometric Transformations

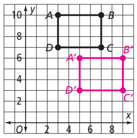
Use matrix addition to find the coordinates of each image after a translation of 2 units right and 4 units down. Then graph each image on the same coordinate plane as its preimage.

1. $A(3, 9)$, $B(2, 7)$, $C(5, 7)$



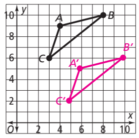
$A'(5, 5)$, $B'(4, 3)$, $C'(7, 3)$

2. $A(3, 10)$, $B(7, 10)$, $C(7, 7)$, $D(3, 7)$



$A'(5, 6)$, $B'(9, 6)$, $C'(9, 3)$, $D'(5, 3)$

3. $A(4, 9)$, $B(8, 10)$, $C(3, 6)$



$A'(6, 5)$, $B'(10, 6)$, $C'(5, 2)$

4. Error Analysis Triangle *ABC* has vertices (2, 2), (6, 3), and (4, 5). Angela translated $\triangle ABC$ 4 units right and 7 units up. She found the coordinates (9, 6), (13, 7), and (11, 9). What error did Angela make? What are the correct coordinates?
Angela increased the *x*-coordinates by 7 and the *y*-coordinates by 4 when she should have increased the *x*-coordinates by 4 and the *y*-coordinates by 7; (6, 9), (10, 10), and (8, 12)

Find the coordinates of each image after the given dilation.

5. $\begin{bmatrix} 3 & 5 & 8 \\ 1 & 6 & 4 \end{bmatrix}, 3$ (9, 3), (15, 18), (24, 12)

6. $\begin{bmatrix} -3 & -6 & -3 & 0 \\ 5 & 0 & -5 & 2 \end{bmatrix}, 2$ (−6, 10), (−12, 0), (−6, −10), (0, 4)

7. $\begin{bmatrix} 6 & 8 & 10 \\ 2 & 6 & 4 \end{bmatrix}, 0.5$ (3, 1), (4, 3), (5, 2)

8. Reasoning Your classmate said that dilating an image by a factor of 0.25 will increase the size of the image. Is he correct? Explain. No, he is not correct. Dilating an image by a factor less than 1 decreases the size of the image. Multiplying by 0.25 is like dividing by 4.

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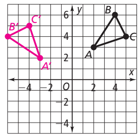
12-5 Practice (continued) *Form K*

Geometric Transformations

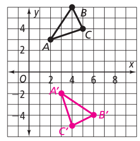
Rotate the triangle with vertices $A(2, 3)$, $B(4, 6)$, $C(5, 4)$ by the given amounts. Write the vertices of the image. Then graph each image on the same coordinate plane as its preimage.

9. 90°



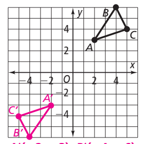
$A'(-3, 2)$, $B'(-6, 4)$, $C'(-4, 5)$

10. 270°



$A'(3, -2)$, $B'(6, -4)$, $C'(4, -5)$

11. 180°



$A'(-2, -3)$, $B'(-4, -6)$, $C'(-5, -4)$

12. Multiple Choice Which of the following rotation matrices represents a rotation of 360°? C

Ⓐ $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Ⓒ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Ⓓ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Find the coordinates of each image after reflection in the given line.

13. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 4 \end{bmatrix}$; $y = x$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 4 \end{bmatrix}$
(2, 1), (5, 3), (4, 5)

14. $\begin{bmatrix} 3 & 1 & 5 \\ 1 & 4 & 5 \end{bmatrix}$; *y*-axis
(−3, 1), (−1, 4), (−5, 5)

15. $\begin{bmatrix} 2 & 8 & 2 & 8 \\ 3 & 3 & 7 & 7 \end{bmatrix}$; $y = -x$
(−3, −2), (−3, −8), (−7, −2), (−7, −8)

16. Writing Explain the difference between reflecting a figure and rotating a figure.
Answers may vary. Sample: Reflecting a figure involves flipping the figure across a given line. Rotating a figure involves turning the figure about a single point.

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12-5 Standardized Test Prep

Geometric Transformations

Multiple Choice

For Exercises 1–3, choose the correct letter.

1. A triangle has vertices $A(4, 6)$, $B(1, -5)$, and $C(-3, 1)$. What are the vertices of the image of the triangle after a rotation of 90°? A

Ⓐ $A'(-6, 4)$, $B'(5, 1)$, $C'(-1, -3)$ Ⓒ $A'(6, 4)$, $B'(-5, 1)$, $C'(1, -3)$

Ⓑ $A'(-4, 6)$, $B'(-1, -5)$, $C'(3, 1)$ Ⓓ $A'(-6, -4)$, $B'(5, -1)$, $C'(-1, 3)$

2. The matrix $\begin{bmatrix} -5 & -3 & 0 & 1 \\ 5 & 6 & 7 & 9 \end{bmatrix}$ represents the vertices of a polygon. Which matrix represents the vertices of the image of the polygon after a dilation of 3? G

Ⓕ $\begin{bmatrix} -5 & -3 & 0 & 1 \\ 8 & 9 & 10 & 12 \end{bmatrix}$ Ⓗ $\begin{bmatrix} -2 & 0 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{bmatrix}$

Ⓖ $\begin{bmatrix} -15 & -9 & 0 & 3 \\ 15 & 18 & 21 & 27 \end{bmatrix}$ Ⓘ $\begin{bmatrix} -15 & -9 & 0 & 3 \\ 5 & 6 & 7 & 9 \end{bmatrix}$

3. The matrix $\begin{bmatrix} -2 & 0 & 1 & 3 & 6 \\ 3 & 4 & 5 & 4 & 2 \end{bmatrix}$ represents the vertices of a polygon. Which matrix represents the vertices of the image of the polygon after a translation 1 unit left and 2 units up? C

Ⓐ $\begin{bmatrix} 0 & 2 & 3 & 5 & 8 \\ 2 & 3 & 4 & 3 & 1 \end{bmatrix}$ Ⓒ $\begin{bmatrix} -3 & -1 & 0 & 2 & 5 \\ 5 & 6 & 7 & 6 & 4 \end{bmatrix}$

Ⓑ $\begin{bmatrix} -1 & 1 & 2 & 4 & 7 \\ 5 & 6 & 7 & 6 & 4 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -3 & 0 & 1 & 3 & 6 \\ 5 & 4 & 5 & 4 & 2 \end{bmatrix}$

Short Response

4. The matrix $\begin{bmatrix} -4 & -6 & -3 & -1 \\ -1 & -2 & -5 & -8 \end{bmatrix}$ represents the vertices of polygon *ABCD*. List the coordinates of the vertices of $A'B'C'D'$ after a reflection across the *y*-axis. Show your work.

[2] $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & -6 & -3 & -1 \\ -1 & -2 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 3 & 1 \\ -1 & -2 & -5 & -8 \end{bmatrix}$
$A'(4, -1)$, $B'(6, -2)$, $C'(3, -5)$, $D'(1, -8)$
[1] incorrect or incomplete work shown
[0] incorrect answers and no work shown OR no answers given

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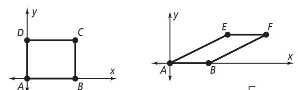
12-5 Enrichment

Geometric Transformations

Rotations, translations, and reflections are called *Euclidean transformations*. They are isometric. The size and shape of a transformed figure does not change, but the location and orientation may change.

Affine transformations are generalizations of these Euclidean transformations. Affine transformations are nonisometric and do not preserve length and angle measure. As a result, the image may be a different shape than the preimage.

One type of affine transformation is a *shear transformation*. A shear transformation takes a shape and "pushes" it in a direction that is parallel to either the *x*-axis or the *y*-axis. For example, in the figures below, the image of rectangle *ABCD* is parallelogram *ABFE* under a shear transformation.



1. The matrix that represents rectangle *ABCD* above is $\begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$. The shear matrix used to produce parallelogram *ABFE* is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The element 2 is the shear factor. Multiply $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ to find the new coordinates of the vertices of parallelogram *ABFE*. What are the new coordinates? (0, 0), (4, 0), (10, 3), (6, 3)

2. How did the shear matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ change rectangle *ABCD*? Answers may vary. Sample: Rectangle *ABCD* was pushed parallel to the *x*-axis to form a parallelogram.

3. Multiply $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$. How does this shear matrix affect rectangle *ABCD*? What are the new coordinates of the vertices of the parallelogram?
Answers may vary. Sample: Rectangle *ABCD* was pushed parallel to the *y*-axis to form a parallelogram. The new coordinates are (0,0), (4, 8), and (4, 11), (0, 3).

4. Write a shear matrix that would push a figure parallel to the *x*-axis.
Answers may vary. Sample: any matrix of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

5. Write a shear matrix that would push a figure parallel to the *y*-axis.
Answers may vary. Sample: any matrix of the form $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

6. Draw a rectangle on a coordinate grid. Use your shear matrices from exercises 4 and 5 to transform the rectangle. Check students' work.

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12-5 Reteaching
Geometric Transformations

- A *translation* changes the position of a geometric figure without changing its size, shape, or orientation.
- You can use matrix addition to find the coordinates of a polygon's vertices after a translation.

Problem

A polygon has vertices $A(0, 0)$, $B(-2, 3)$, $C(-5, 3)$, and $D(-5, 0)$. If you translate it 5 units to the right and 3 units down, what are the coordinates of the vertices of its image $A'B'C'D'$? Draw $ABCD$ and its image.

$$\begin{array}{r c} & \begin{array}{cccc} A & B & C & D \end{array} \\ x\text{-coordinates} \rightarrow & \left[\begin{array}{cccc} 0 & -2 & -5 & -5 \\ 0 & 3 & 3 & 0 \end{array}\right] \\ y\text{-coordinates} \rightarrow & \end{array}$$

Write the vertices of $ABCD$ as a matrix.

$$\begin{bmatrix} 5 & 5 & 5 & 5 \\ -3 & -3 & -3 & -3 \end{bmatrix}$$

Write the translation matrix. The first row represents moving 5 units right. The second row represents moving 3 units down.

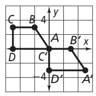
$$\begin{bmatrix} 0 & -2 & -5 & -5 \\ 0 & 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 & 5 \\ -3 & -3 & -3 & -3 \end{bmatrix}$$

Add the translation matrix to the $ABCD$ matrix.

$$\begin{array}{cc} & \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ = & \begin{bmatrix} 5 & 3 & 0 & 0 \\ -3 & 0 & 0 & -3 \end{bmatrix} \end{array}$$

The sum represents the vertices of $A'B'C'D'$:

$A'(5, -3)$, $B'(3, 0)$, $C'(0, 0)$, and $D'(0, -3)$.



Draw $ABCD$ and $A'B'C'D'$. Label each vertex.

Exercises

A polygon has vertices $A(0, 0)$, $B(-2, 3)$, $C(1, 4)$, and $D(3, 2)$. Write a matrix to represent the vertices of $A'B'C'D'$ after each translation.

1. 1 unit right, 2 units down $\begin{bmatrix} 1 & -1 & 2 & 4 \\ -2 & 1 & 2 & 0 \end{bmatrix}$
2. 3 units left, 1 unit down $\begin{bmatrix} -3 & -5 & -2 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix}$
3. 4 units left, 3 units up $\begin{bmatrix} -4 & -6 & -3 & -1 \\ 3 & 6 & 7 & 5 \end{bmatrix}$
4. 2 units right, 4 units up $\begin{bmatrix} 2 & 0 & 3 & 5 \\ 4 & 7 & 8 & 6 \end{bmatrix}$

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12-5 Reteaching (continued)
Geometric Transformations

- You can use scalar multiplication to find the coordinates of a polygon's vertices after a dilation.
- You can use matrix multiplication to find the coordinates of a polygon's vertices after a rotation or reflection.

Problem

A polygon has vertices $A(0, 0)$, $B(-2, 3)$, $C(-5, 3)$, and $D(-5, 0)$. What are the vertices of the polygon after a dilation of 3?

$$\begin{array}{r} x\text{-coordinates} \rightarrow \\ y\text{-coordinates} \rightarrow \end{array} \begin{bmatrix} 0 & -2 & -5 & -5 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

Write the vertices of $ABCD$ as a matrix.

$$A'B'C'D' = (3)\begin{bmatrix} 0 & -2 & -5 & -5 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

Multiply by the dilation factor.

$$= \begin{bmatrix} 0 & -6 & -15 & -15 \\ 0 & 9 & 9 & 0 \end{bmatrix}$$

The product represents the vertices of $A'B'C'D'$:
$A'(0, 0)$, $B'(-6, 9)$, $C'(-15, 9)$, and $D'(-15, 0)$.

Problem

A polygon has vertices $A(0, 0)$, $B(-2, 3)$, $C(-5, 3)$, and $D(-5, 0)$. What are the vertices of the polygon after a rotation of 90°?

$$\begin{array}{r} x\text{-coordinates} \rightarrow \\ y\text{-coordinates} \rightarrow \end{array} \begin{bmatrix} 0 & -2 & -5 & -5 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

Write the vertices of $ABCD$ as a matrix.

$$A'B'C'D' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 & -5 & -5 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

Multiply by the appropriate rotation matrix.

$$= \begin{bmatrix} 0 & -3 & -3 & 0 \\ 0 & -2 & -5 & -5 \end{bmatrix}$$

The product represents the vertices of $A'B'C'D'$:
$A'(0, 0)$, $B'(-3, -2)$, $C'(-3, -5)$, and $D'(0, -5)$.

Exercises

A polygon has vertices $A(0, 0)$, $B(-2, 3)$, $C(1, 4)$, and $D(3, 2)$. Write a matrix to represent the vertices of $A'B'C'D'$ after each transformation.

5. dilation of 0.5 $\begin{bmatrix} 0 & -1 & 0.5 & 1.5 \\ 0 & 1.5 & 2 & 1 \end{bmatrix}$
6. rotation of 180° $\begin{bmatrix} 0 & 2 & -1 & -3 \\ 0 & -3 & -4 & -2 \end{bmatrix}$
7. reflection across y-axis $\begin{bmatrix} 0 & 2 & -1 & -3 \\ 0 & 3 & 4 & 2 \end{bmatrix}$
8. reflection across $y = x$ $\begin{bmatrix} 0 & 3 & 4 & 2 \\ 0 & -2 & 1 & 3 \end{bmatrix}$

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12-6 ELL Support
Vectors

Choose the word from the list that best matches each sentence.

dot product	initial point	magnitude
normal vectors	terminal point	vector

1. A vecor begins at the initial point.
2. The dot product of vectors $\langle v_1, v_2\rangle$ and $\langle w_1, w_2\rangle$ is $v_1w_1 + v_2w_2$.
3. A vector has both distance and direction.
4. Normal vectors are perpendicular to each other.
5. A vector ends at the terminal point.
6. The magnitude of a vector is the length of the arrow.

Circle the dot product of the following vectors. Then tell whether the vectors are normal.

7. $\mathbf{t} = \langle 3, 1\rangle$, $\mathbf{u} = \langle 2, 5\rangle$ (A) 13 **(B) 11** (C) 0
   not normal
8. $\mathbf{v} = \langle -2, 4\rangle$, $\mathbf{w} = \langle 6, 3\rangle$ (A) −26 (B) −24 **(C) 0**
   normal
9. $\mathbf{m} = \langle 3, 6\rangle$, $\mathbf{n} = \langle -8, 4\rangle$ (A) −48 (B) −14 **(C) 0**
   normal
10. $\mathbf{a} = \langle -2, 7\rangle$, $\mathbf{b} = \langle 6, 4\rangle$ **(A) 16** (B) 10 (C) 0
   not normal

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12-6 Think About a Plan
Vectors

Aviation A twin-engine airplane has a speed of 300 mi/h in still air. Suppose the airplane heads south and encounters a wind blowing 50 mi/h due east. What is the resultant speed of the airplane?

Know

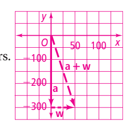
1. The airplane is traveling south at a speed of 300 mi/h.
2. The wind is blowing east at a speed of 50 mi/h.

Need

3. To solve the problem I need to find: the sum of the vectors that represent the speed of the airplane and the speed of the wind.

Plan

4. Sketch the speed of the airplane and the speed of the wind as vectors. Then use the tip-to-tail method to sketch $\mathbf{a} + \mathbf{w}$.



5. What is the component form of the vector for the speed of the airplane? $\mathbf{a} =$ $\langle 0, -300\rangle$
6. What is the component form of the vector for the speed of the wind? $\mathbf{w} =$ $\langle 50, 0\rangle$
7. Express $\mathbf{a} + \mathbf{w}$ in component form. $\mathbf{a} + \mathbf{w} =$ $\langle 50, -300\rangle$
8. What equation can you use to find the magnitude of $\mathbf{a} + \mathbf{w}$? $|\mathbf{a} + \mathbf{w}| =$ $\sqrt{(50)^2 + (-300)^2}$
9. What is the resultant speed of the airplane? about 304 mi/h in a S/SE direction

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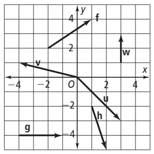
12-6 Practice — Form G
Vectors

Referring to the graph, what are the component forms of the following vectors?



1. u ⟨3, −3⟩
2. v ⟨−4, 1⟩
3. w ⟨0, 2⟩
4. f ⟨3, 2⟩
5. g ⟨3, 0⟩
6. h ⟨1, −3⟩

Transform each vector as described. Write the resultant vector in component form.

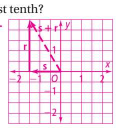
7. ⟨2, −8⟩; rotate 270° ⟨−8, −2⟩
8. ⟨6, 4⟩; rotate 90° ⟨−4, 6⟩
9. ⟨−9, −3⟩; reflect across y-axis ⟨9, −3⟩
10. ⟨−3, 7⟩; reflect across x-axis ⟨−3, −7⟩
11. ⟨0, 16⟩; reflect across $y = -x$ ⟨−16, 0⟩
12. ⟨5, 2⟩; reflect across $y = x$ ⟨2, 5⟩

Let u = ⟨3, −4⟩, v = ⟨−7, 8⟩, and w = ⟨5, 5⟩. Find the component forms of the following vectors.

13. u + w ⟨8, 1⟩
14. w + v ⟨−2, 13⟩
15. v − w ⟨−12, 3⟩
16. −2v ⟨14, −16⟩
17. 3u ⟨9, −12⟩
18. $\frac{3}{5}$w ⟨3, 3⟩

Determine whether the vectors in each pair are normal.

19. ⟨12, −4⟩ and ⟨3, 9⟩ yes
20. ⟨7, 11⟩ and ⟨11, 7⟩ no
21. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$ no
22. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ yes
23. The speed of a swimmer in still water is 1.5 mi/h. The swimmer swims due west in a current flowing due north at 2.5 mi/h.
   a. Use the tip-to-tail method to draw a vector representing the speed and direction of the swimmer.
   b. Write a formula to find the swimmer's speed. $|s + r| = \sqrt{s^2 + r^2}$
   c. What is the swimmer's approximate speed, rounded to the nearest tenth?
   about 2.9 mi/h

23a.


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12-6 Practice (continued) — Form G
Vectors

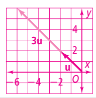
Let u = $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$, v = $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$, and w = $\begin{bmatrix} -2 \\ 6 \end{bmatrix}$. Find the following vectors.

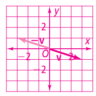
24. 3u + 2w $\begin{bmatrix} -19 \\ 39 \end{bmatrix}$
25. −v + 3w $\begin{bmatrix} -10 \\ 14 \end{bmatrix}$
26. w − $\frac{3}{2}$v − 2u $\begin{bmatrix} 2 \\ -18 \end{bmatrix}$
27. A bird flies 16 mi/h in still air. Suppose the bird flies due south with a wind blowing 15 mi/h due east. What is the resultant speed of the bird rounded to the nearest mile per hour? about 22 mi/h
28. A model rocket lands 245 ft west and 162 ft south of the point from which it was launched. How far did the rocket fly? Round your answer to the nearest foot.
   about 294 ft
29. Consider a polygon with vertices at $A(-3, 5)$, $B(2, 3)$, $C(4, -4)$, and $D(-6, -3)$. Express the sides of the polygon as vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$.
   $\overrightarrow{AB}$ = ⟨5, −2⟩, $\overrightarrow{BC}$ = ⟨2, −7⟩, $\overrightarrow{CD}$ = ⟨−10, 1⟩, $\overrightarrow{DA}$ = ⟨3, 8⟩

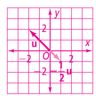
Let a = ⟨7, 5⟩, b = ⟨4, −1⟩, and c = ⟨0, −3⟩. Solve each of the following for the unknown vector v.

30. b − v = c ⟨4, 2⟩
31. v + a = b ⟨−3, −6⟩
32. a + b = v − c ⟨11, 1⟩
33. a − v + b − c = ⟨1, 1⟩ ⟨10, 6⟩
34. A train leaves Dawson station and travels 360 mi due north. Then it turns and travels 120 mi due west to reach New Port. If the train travels 75 mi/h on a straight route directly back to Dawson, how long will the return trip take? Round your answer to the nearest hour.
   about 5 h
35. **Reasoning** Identify the additive identity vector v, if it exists. Explain your reasoning.
   $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$; if $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, then $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Let u = $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and v = $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Graph the following vectors.

36. 3u


37. −v


38. $-\frac{1}{2}$u


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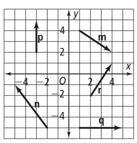
12-6 Practice — Form K
Vectors

Use the graph at the right. What are the component forms of the following vectors?



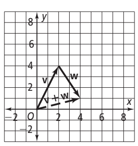
1. m
   initial point = (1, 4)
   terminal point = (4, 2)
   (1 − 1, 4 − 4) = (0, 0)
   (4 − 1, 2 − 4) = (3, −2)
   m = $\boxed{⟨3, −2⟩}$
2. n ⟨−3, 4⟩
3. p ⟨0, 3⟩
4. q ⟨4, 0⟩
5. r ⟨2, 3⟩

Transform each vector as described. Write the resulting vectors in component form.

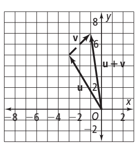
6. ⟨4, 2⟩; rotate 180°
   $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$
   ⟨−4, −2⟩
7. ⟨−3, 7⟩; rotate 90°
   ⟨−7, −3⟩
8. ⟨−5, −8⟩; rotate 270°
   ⟨−8, 5⟩

Find the sums of the following vectors. Round each sum to the nearest hundredth. Use the coordinate grids to help you solve the problems.

9. v = ⟨2, 4⟩, w = ⟨2, −3⟩
   $|v + w| = \sqrt{4^2 + 1^2}$
   v + w ≈ $\boxed{4.12}$



10. u = ⟨−3, 5⟩, v = ⟨2, 2⟩
   ≈ 7.07



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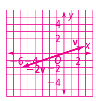
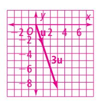
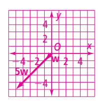
12-6 Practice (continued) — Form K
Vectors

Let u = ⟨1, −3⟩, v = ⟨3, 1⟩, and w = ⟨−1, −1⟩. Find the component forms of the following vectors. Then graph each vector on the coordinate plane.

11. −2v
   −2 · ⟨3, 1⟩
   ⟨−2(3), −2(1)⟩
   ⟨−6, −2⟩
12. 3u
   ⟨3, −9⟩
13. 5w
   ⟨−5, −5⟩







14. **Reasoning** You graphed the vector 4v. Your friend graphed the vector −4v. Whose vector will have the greater magnitude? Explain. Both vectors will have the same magnitude, but they will be pointing in opposite directions because one is positive and the other is negative.

Find the dot products of the following vectors.

15. t = ⟨4, 7⟩, u = ⟨−2, 5⟩
   t · u = (4)(−2) + (7)(5)
   = −8 + 35
   = $\boxed{27}$
16. v = ⟨2, 6⟩, w = ⟨5, 3⟩
   v · w = $\boxed{28}$
17. a = ⟨7, −1⟩, b = ⟨1, 7⟩
   a · b = $\boxed{0}$
18. **Multiple Choice** Which of the following pairs of vectors is normal? C
   (A) ⟨2, 5⟩, ⟨3, 8⟩ (B) ⟨−5, 7⟩, ⟨−8, 1⟩ (C) ⟨3, 2⟩, ⟨−4, 6⟩ (D) ⟨1, 1⟩, ⟨2, 0⟩

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12-6 Standardized Test Prep
Vectors

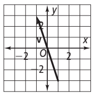
Multiple Choice

For Exercises 1–5, choose the correct letter.

1. Let $\mathbf{u} = \langle 4, -7\rangle$ and $\mathbf{v} = \langle -1, 3\rangle$. What is $|\mathbf{u} + \mathbf{v}|$? **D**

Ⓐ -5 Ⓑ -1 Ⓒ 1 Ⓓ 5

2. What is the component form of vector **v**? **F**



Ⓕ $\langle -2, 6\rangle$ Ⓗ $\langle -1, 3\rangle$
Ⓖ $\langle 0, 0\rangle$ Ⓘ $\langle 1, -3\rangle$

3. Which represents the vector $\mathbf{u} = \langle 16, -9\rangle$ rotated 180°? **C**

Ⓐ $\begin{bmatrix} -9 \\ -16 \end{bmatrix}$ Ⓑ $\begin{bmatrix} -9 \\ 16 \end{bmatrix}$ Ⓒ $\begin{bmatrix} -16 \\ 9 \end{bmatrix}$ Ⓓ $\begin{bmatrix} 16 \\ 9 \end{bmatrix}$

4. Let $\mathbf{u} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$. What is the vector $-2\mathbf{u} - \frac{5}{2}\mathbf{v}$? **G**

Ⓕ $\begin{bmatrix} -15 \\ -26 \end{bmatrix}$ Ⓖ $\begin{bmatrix} -5 \\ -6 \end{bmatrix}$ Ⓗ $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$ Ⓘ $\begin{bmatrix} 10 \\ 4 \end{bmatrix}$

5. Which represents the vector $\mathbf{w} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$ reflected across $y = x$? **B**

Ⓐ $\begin{bmatrix} -14 \\ 8 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 14 \\ 8 \end{bmatrix}$ Ⓒ $\begin{bmatrix} -8 \\ 14 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -8 \\ -14 \end{bmatrix}$

Short Response

6. Let $\mathbf{p} = \langle 6, -1\rangle$ and $\mathbf{q} = \langle 3, 5\rangle$. Are **p** and **q** normal? Show your work.

[2] $\mathbf{p} \cdot \mathbf{q} = (6)(3) + (-1)(5) = 18 - 5 = 13$
The dot product does not equal zero. The vectors are not normal.
[1] incorrect or incomplete work shown
[0] incorrect answer and no work shown OR no answer given

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12-6 Enrichment
Vectors

A unit vector, **u**, is a vector of length 1. Any vector **v** can be made into a unit vector by dividing by its length. This is called *normalizing the vector*.

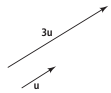
$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

1. The first step in normalizing a vector is to find its magnitude, $|v|$. Find the magnitude of vector $\langle 3, 4\rangle$.
$\sqrt{3^2 + 4^2} = 5$

2. Next evaluate $\frac{1}{5}\langle 3, 4\rangle$ to determine the unit vector.
$\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

3. Normalize the vector $\langle -2, 5\rangle$.
$\left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$ or $\left\langle \frac{-2\sqrt{29}}{29}, \frac{5\sqrt{29}}{29} \right\rangle$

Any vector can be thought of as being a multiple of a unit vector. For example, the vector below is 3 times the length of the unit vector in the same direction.



4. Given the unit vector $\left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$, which vector of the same direction has a magnitude of 10 units?
$\langle 5, -5\sqrt{3}\rangle$

5. Normalize the vector $\langle -4, -5\rangle$.
$\left\langle \frac{-4}{\sqrt{41}}, \frac{-5}{\sqrt{41}} \right\rangle$ or $\left\langle \frac{-4\sqrt{41}}{41}, \frac{-5\sqrt{41}}{41} \right\rangle$

6. Which vector is 4 times the magnitude of the unit vector in Exercise 5?
$\left\langle \frac{-16}{\sqrt{41}}, \frac{-20}{\sqrt{41}} \right\rangle$ or $\left\langle \frac{-16\sqrt{41}}{41}, \frac{-20\sqrt{41}}{41} \right\rangle$

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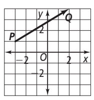
12-6 Reteaching
Vectors

A vector shows *magnitude* (length) and *direction*, but does NOT represent location. You can name a vector using its initial and terminal points, or you can write it in *component form*. The component form gives you the coordinates of the terminal point when the initial point is translated to the origin.

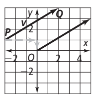


Problem

What is the component form of vector $\mathbf{v} = \overrightarrow{PQ}$ in the graph above?

Step 1 Translate the vector so that the initial point becomes (0, 0). The translation is 3 units right and 1 unit down.

Step 2 Find the coordinates of the translated terminal point. The original terminal point was (2, 4). The translated terminal point is $(2 + 3, 4 - 1) = (5, 3)$



The component form is $\mathbf{v} = \langle 5, 3\rangle$.

You can add, subtract, and do scalar multiplication with vectors in component form.

Problem

Let $\mathbf{u} = \langle 5, -8\rangle$ and $\mathbf{v} = \langle 3, 1\rangle$. What is $2\mathbf{u} + \mathbf{v}$ in component form?

$2\mathbf{u} = 2\langle 5, -8\rangle$	Substitute the component form for **u**.
$= \langle (2)(5), (2)(-8)\rangle = \langle 10, -16\rangle$	Multiply each coordinate by the scalar 2.
$2\mathbf{u} + \mathbf{v} = \langle 10, -16\rangle + \langle 3, 1\rangle$	Substitute the component forms for **2u** and **v**.
$= \langle (10 + 3), (-16 + 1)\rangle = \langle 13, -15\rangle$	Add the *x*-coordinates. Add the *y*-coordinates.

The component form is $2\mathbf{u} + \mathbf{v} = \langle 13, -15\rangle$.

Exercises

Let $\mathbf{r} = \langle -6, 3\rangle$, $\mathbf{s} = \langle 4, 7\rangle$, and $\mathbf{t} = \langle 2, -5\rangle$. Find the component forms of the following vectors.

1. $\mathbf{r} - \mathbf{s}$ $\langle -10, -4\rangle$ **2.** $3\mathbf{t} + 2\mathbf{r}$ $\langle -6, -9\rangle$ **3.** $\frac{2}{3}\mathbf{r} - 4\mathbf{s} + \mathbf{t}$ $\langle -18, -31\rangle$

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12-6 Reteaching (continued)
Vectors

If vectors are *normal*, or perpendicular, then their *dot product* equals zero. For vectors $\mathbf{v} = \langle v_1, v_2\rangle$ and $\mathbf{w} = \langle w_1, w_2\rangle$, the dot product is $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2$.

Problem

Are the vectors $\mathbf{r} = \langle 4, 8\rangle$ and $\mathbf{s} = \langle -6, 3\rangle$ normal?

$\mathbf{r} \cdot \mathbf{s} = (4)(-6) + (8)(3)$	Find the dot product.
$= -24 + 24 = 0$	The dot product equals zero.

The vectors are normal.

If a vector is in matrix form, then you can rotate or reflect the vector by multiplying by the appropriate rotation or reflection matrix. For a vector with an initial point (0, 0) and terminal point (v_1, v_2):

Component Form	Matrix Form
$\mathbf{v} = \langle v_1, v_2\rangle$	$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Problem

Rotate the vector $\mathbf{w} = \langle 7, -2\rangle$ by 90°.

$\mathbf{w} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$	Write the vector in matrix form.
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$	Multiply by the 90° rotation matrix.

The rotated vector, written in component form, is $\langle 2, 7\rangle$.

Exercises

Determine whether the vectors in each pair are normal.

4. $\mathbf{p} = \langle 3, -2\rangle$ and $\mathbf{q} = \langle 4, 6\rangle$ **yes** **5.** $\mathbf{u} = \langle 2, 2\rangle$ and $\mathbf{v} = \langle 4, -1\rangle$ **no**

Transform each vector as described. Write your answer in component form.

6. $\langle 3, 10\rangle$; reflect across $y = x$ $\langle 10, 3\rangle$ **7.** $\langle -4, -9\rangle$; rotate 270° $\langle -9, 4\rangle$

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Chapter 12 Quiz 1 — *Form G*

Lessons 12-1 through 12-3

Do you know HOW?

Solve each matrix equation.

1. $\begin{bmatrix} 3 & -8 & 0 \\ 8 & 2 & 1 \end{bmatrix} + X = \begin{bmatrix} 5 & 3 & -9 \\ 2 & 12 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & 11 & -9 \\ -6 & 10 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -4 \\ -3 & 3 \\ 6 & -6 \end{bmatrix} - X = \begin{bmatrix} 5 & 5 \\ 5 & 5 \\ 5 & 5 \end{bmatrix}$ $\begin{bmatrix} -1 & -9 \\ -8 & -2 \\ 1 & -11 \end{bmatrix}$

Use matrices A and B. Find each product, sum, or difference.

$$A = \begin{bmatrix} -1 & 4 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -5 & 1 \\ 0 & 3 \end{bmatrix}$$

3. $-A$ $\begin{bmatrix} 1 & -4 \\ -3 & -2 \end{bmatrix}$

4. $3B$ $\begin{bmatrix} -15 & 3 \\ 0 & 9 \end{bmatrix}$

5. $2A - 4B$ $\begin{bmatrix} 18 & 4 \\ 6 & -8 \end{bmatrix}$

6. AB $\begin{bmatrix} 5 & 11 \\ -15 & 9 \end{bmatrix}$

7. $B \cdot B$ $\begin{bmatrix} 25 & -2 \\ 0 & 9 \end{bmatrix}$

8. $A - BA$ $\begin{bmatrix} -9 & 22 \\ -6 & -4 \end{bmatrix}$

For each matrix, find the determinant. Then find the inverse, if it exists.

9. $\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$ 20; $\begin{bmatrix} \frac{1}{20} & -\frac{3}{20} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ 3; $\begin{bmatrix} 1 & 0 & \frac{2}{3} \\ -1 & -1 & -1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$

11. $\begin{bmatrix} 5 & 10 \\ 4 & 8 \end{bmatrix}$ 0; no inverse exists

12. $\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$ -1; $\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

Do you UNDERSTAND?

13. **Writing** Describe how to find the dimensions of the matrix $\begin{bmatrix} 1 & 3 & 5 & 2 \\ 9 & 8 & 2 & 0 \end{bmatrix}$.

    The dimensions of a matrix are *number of rows* × *number of columns*, so the dimensions of this matrix are 2 × 4.

14. **Error Analysis** A student says that matrix multiplication is commutative for matrices with the same dimensions. Provide a counter-example to show the student's error. any pair of nonzero matrices that are not inverses

15. **Reasoning** For what values of x will the matrix $\begin{bmatrix} 2 & -3 \\ x & 6 \end{bmatrix}$ have an inverse? $x \neq -4$

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Chapter 12 Quiz 2 — *Form G*

Lessons 12-4 through 12-6

Do you know HOW?

Write each system as a matrix equation. Use the matrix equation to solve the system.

1. $\begin{cases} x + 3y = -4 \\ -2x - y = -7 \end{cases}$ $\begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$; (5, −3)

2. $\begin{cases} 2x + y = 1 \\ -x + 3y = -25 \end{cases}$ $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -25 \end{bmatrix}$; (4, −7)

3. $\begin{cases} 3x - 2y = -8 \\ x + 5y = 3 \end{cases}$ $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$; (−2, 1)

Use matrices to perform the following transformations on the triangle with vertices $A(-2, 5)$, $B(-1, -4)$, and $C(3, 2)$. State the coordinates of the vertices of the image.

4. Enlarge by a factor of 3.
   (−6, 15), (−3, −12), (9, 6)

5. Rotate 90°.
   (−5, −2), (4, −1), (−2, 3)

6. Reflect across x-axis.
   (−2, −5), (−1, 4), (3, −2)

7. Reduce by a factor of $\frac{1}{4}$.
   $\left(-\frac{1}{2}, \frac{5}{4}\right), \left(-\frac{1}{4}, -1\right); \left(\frac{3}{4}, \frac{1}{2}\right)$

Let $J = (2, 3)$, $K = (-3, 1)$, $L = (-1, -3)$, and $M = (3, 1)$. What are the component forms of the following vectors?

8. $\overrightarrow{JK}$ ⟨−5, −2⟩

9. $\overrightarrow{JK} + \overrightarrow{JM}$ ⟨−1, 2⟩

10. $2\overrightarrow{ML}$ ⟨−8, −8⟩

Do you UNDERSTAND?

11. **Error Analysis** A student writes a linear system as a matrix equation and finds that the determinant of the coefficient matrix is zero. He says that the solution of the system must be (0, 0). Explain the student's error. What does a determinant of zero tell you about the linear system? Answers may vary. Sample: The student used the determinant as the solution; the system has no solution or infinite solutions.

12. **Reasoning** The matrix $\begin{bmatrix} 0 & n & n & 0 \\ n & n & 0 & 0 \end{bmatrix}$ represents the vertices of square $ABCD$.

    Describe the transformation performed by multiplying the matrix by $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

    The square will be reflected across $y = -x$, and the vertices of square *ABCD* are (0, 0), $(-n, 0)$, $(0, -n)$, and $(-n, -n)$.

13. **Reasoning** Given vector $\overrightarrow{AB}$ with $A(m, n)$ and $B(-m, 0)$, write $\overrightarrow{AB}$ in component form. $\langle -2m, -n \rangle$

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Chapter 12 Chapter Test — *Form G*

Do you know HOW?

Find each sum or difference.

1. $\begin{bmatrix} 9 & 2 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} 7 & -5 \\ 6 & 3 \end{bmatrix}$ $\begin{bmatrix} 2 & 7 \\ 2 & -4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 13 & 16 \\ 24 & -3 & 19 \\ 9 & 10 & 20 \end{bmatrix} + \begin{bmatrix} 22 & 7 & -18 \\ 5 & 15 & 11 \\ 12 & 14 & -17 \end{bmatrix}$ $\begin{bmatrix} 23 & 20 & -2 \\ 29 & 12 & 30 \\ 21 & 24 & 3 \end{bmatrix}$

Find each product if it exists. If it does not exist, write *undefined*.

3. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$ $\begin{bmatrix} 5 & 14 \\ 6 & 12 \\ 18 & 52 \end{bmatrix}$

4. $-3\begin{bmatrix} 3 & 10 \\ 21 & -4 \end{bmatrix}$ $\begin{bmatrix} -9 & -30 \\ -63 & 12 \end{bmatrix}$

5. $\begin{bmatrix} 9 & 15 & 6 \\ -8 & 2 & 7 \\ 63 & -8 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 0 \\ -5 & 0 & 8 \end{bmatrix}$ undefined

6. $\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$ $\begin{bmatrix} 8 & 3 \\ 0 & 0 \end{bmatrix}$

Use matrices to perform the following transformations on the triangle with vertices $A(1, 1)$, $B(3, 4)$, and $C(0, -5)$. State the coordinates of the vertices of the image.

7. a dilation of $\frac{1}{2}$ $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, 2\right), \left(0, -\frac{5}{2}\right)$

8. a rotation of 180° (−1, −1), (−3, −4), (0, 5)

9. a reflection across $y = x$
   (1, 1), (4, 3), (−5, 0)

10. a reflection across x-axis
    (1, −1), (3, −4), (0, 5)

11. Let $P = (5, 3)$ and $Q = (-2, -4)$. Write the vector $\overrightarrow{QP}$ in component form. ⟨7, 7⟩

Find the determinant of each matrix.

12. $\begin{bmatrix} -6 & -7 \\ 5 & 8 \end{bmatrix}$ -13

13. $\begin{bmatrix} 1 & -5 \\ -2 & 9 \end{bmatrix}$ -1

14. $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 0 & 5 \\ 0 & 4 & 0 \end{bmatrix}$ 52

Find the inverse of each matrix, if it exists.

15. $\begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}$ $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 1 \end{bmatrix}$

16. $\begin{bmatrix} 7 & 8 \\ -14 & -16 \end{bmatrix}$ no inverse

17. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & -1 & 4 \\ 1 & 2 & -5 \\ 0 & -1 & 2 \end{bmatrix}$

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Chapter 12 Chapter Test (continued) — *Form G*

Solve each matrix equation.

18. $\begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}X = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\begin{bmatrix} -11 \\ 9 \end{bmatrix}$

19. $\begin{bmatrix} 6 & 2 \\ -1 & 4 \end{bmatrix} - X = \begin{bmatrix} 8 & 1 \\ 1 & 3 \end{bmatrix}$ $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$

20. $\begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}X = \begin{bmatrix} 8 & -1 \\ 6 & 0 \end{bmatrix}$ $\begin{bmatrix} -20 & 1 \\ 26 & -1 \end{bmatrix}$

21. $2X = \frac{1}{4}\begin{bmatrix} -6 & 2 \\ 8 & -8 \end{bmatrix}$ $\begin{bmatrix} -\frac{3}{4} & \frac{1}{4} \\ 1 & -1 \end{bmatrix}$

Solve each system of equations using a matrix equation.

22. $\begin{cases} 2x - y = 2 \\ -3x + 2y = -4 \end{cases}$ (0, −2)

23. $\begin{cases} 2x + 3y + 4z = 3 \\ -13x + 5y - 2z = 3 \\ -3x + 4y + 3z = 6 \end{cases}$ (−2, −3, 4)

24. An airplane has a speed of 240 mi/h in still air. The plane flies due south with the wind blowing 30 mi/h due west. What is the resultant speed of the plane, rounded to the nearest whole mi/h? about 242 mi/h

Do you UNDERSTAND?

25. **Reasoning** Are $\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$ inverses of each other? Show why or why not.
    Yes; the product of a 2 × 2 matrix and its inverse is the identity matrix.
    $\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -2 + 3 & -2 + 2 \\ 3 + (-3) & 3 + (-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

26. **Error Analysis** A student says that
    $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Explain and correct the student's error. Answers may vary. Sample: Repeated matrix addition is equivalent to scalar multiplication; $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 4\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

27. **Open-Ended** Write a pair of vectors that are normal. Use a dot product to show your answer is correct. Answers may vary. Sample: $\mathbf{v} = \langle 1, 2 \rangle$, $\mathbf{w} = \langle -6, 3 \rangle$; $\mathbf{v} \cdot \mathbf{w} = (1)(-6) + (2)(3) = -6 + 6 = 0$

28. **Writing** Explain how to find the determinant of a 2 × 2 matrix.
    Answers may vary. Sample: For the matrix $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, the determinant is the difference of a_1a_4 and a_2a_3.

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Chapter 12 Quiz 1 — *Form K*

Lessons 12–1 and 12–2

Do you know HOW?

Find each sum or difference.

1. $\begin{bmatrix} 2 & 7 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 9 & -5 \end{bmatrix}$ $\begin{bmatrix} 6 & 5 \\ 9 & -2 \end{bmatrix}$

2. $\begin{bmatrix} -1 & 4 \\ 5 & -2 \\ 3 & 8 \end{bmatrix} + \begin{bmatrix} 6 & -9 \\ 7 & 3 \\ -3 & 2 \end{bmatrix}$ $\begin{bmatrix} 5 & -5 \\ 12 & 1 \\ 0 & 10 \end{bmatrix}$

3. $\begin{bmatrix} 7 & 1 \\ -4 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 4 & -4 \\ -5 & 3 \end{bmatrix}$

Find the value of each variable.

4. $\begin{bmatrix} 6 & 9 \\ 2 & x \end{bmatrix} + \begin{bmatrix} y & 2 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 11 \\ 7 & 4 \end{bmatrix}$ $x = -3; y = 3$

5. $\begin{bmatrix} 2x+1 & -8 \\ 5 & 3y \end{bmatrix} = \begin{bmatrix} 9 & 4z \\ 5 & 6 \end{bmatrix}$ $x = 4; y = 2; z = -2$

Solve each matrix equation.

6. $\begin{bmatrix} 7 & 5 \\ 1 & -3 \end{bmatrix} + 2X = \begin{bmatrix} 13 & 1 \\ 9 & -15 \end{bmatrix}$ $\begin{bmatrix} 3 & -2 \\ 4 & -6 \end{bmatrix}$

7. $\begin{bmatrix} -2 & 3 \\ 8 & -5 \end{bmatrix} - \frac{1}{4}X = \begin{bmatrix} -1 & 4 \\ 6 & -2 \end{bmatrix}$ $\begin{bmatrix} -4 & -4 \\ 8 & -12 \end{bmatrix}$

Do you UNDERSTAND?

8. **Writing** Describe the process used to multiply matrices. Then find the product of $\begin{bmatrix} 3 & 8 \\ 7 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & -2 \\ 0 & 3 \end{bmatrix}$. **First, multiply the elements in the first row of A by the elements in the first column of B and find the sum of the products. Then find the sum of the product of the elements in the first row of A and the elements in the second column of B. Next, multiply the second row of A by the first column of B and add. Finally, multiply the second row of A by the second column of B and add; $\begin{bmatrix} 15 & 18 \\ 35 & -11 \end{bmatrix}$**

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Chapter 12 Quiz 2 — *Form K*

Lessons 12–3 and 12–4

Do you know HOW?

Evaluate the determinant of each matrix.

1. $\begin{bmatrix} 9 & 5 \\ 3 & 2 \end{bmatrix}$ 3

2. $\begin{bmatrix} 5 & -2 & 6 \\ -1 & 2 & 1 \\ 3 & 7 & 4 \end{bmatrix}$ -87

3. $\begin{bmatrix} -4 & 2 \\ 7 & 8 \end{bmatrix}$ -46

Find the inverse of each matrix, if one exists.

4. $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

5. $\begin{bmatrix} 10 & 3 \\ 7 & 2 \end{bmatrix}$ $\begin{bmatrix} -2 & 3 \\ 7 & -10 \end{bmatrix}$

6. $\begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix}$ $\begin{bmatrix} \frac{3}{2} & 2 \\ 2 & 3 \end{bmatrix}$

Write each system as a matrix equation.

7. $\begin{cases} 4x - y = 3 \\ 2x + 3y = 19 \end{cases}$ $\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \end{bmatrix}$

8. $\begin{cases} 3x = 12 - 3y \\ y = 2x - 14 \end{cases}$ $\begin{bmatrix} 3 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -14 \end{bmatrix}$

9. $\begin{cases} x + y + z = 6 \\ 4x = 4 + 2z \\ 3y = 1 - z \end{cases}$ $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & -2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$

Do you UNDERSTAND?

10. **Error Analysis** Your classmate solved the system of equations $\begin{cases} 5x = 6 - 3y \\ x + 2y = 11 \end{cases}$. He used the matrix equation below. What error did he make? What are the values of x and y?

$$\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$$

He did not rewrite the system so that the variables were in the same order in each equation; $x = -3$ and $y = 7$

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Chapter 12 Test — *Form K*

Do you know HOW?

Find each sum or difference.

1. $\begin{bmatrix} 6 & 3 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -7 \\ 5 & -2 \end{bmatrix}$ $\begin{bmatrix} 7 & -4 \\ 4 & -6 \end{bmatrix}$

2. $\begin{bmatrix} 9 & 4 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ -2 & 6 \end{bmatrix}$ $\begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ -4 & 1 \\ 8 & -6 \end{bmatrix} + \begin{bmatrix} 5 & -6 \\ -5 & 7 \\ 2 & 3 \end{bmatrix}$ $\begin{bmatrix} 7 & -3 \\ -9 & 8 \\ 10 & -3 \end{bmatrix}$

Find the value of each variable.

4. $\begin{bmatrix} 3 & x \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} -1 & -4 \\ y & z \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 9 & 5 \end{bmatrix}$ $x = 1; y = 3; z = -2$

5. $\begin{bmatrix} 2x+4 & 10 \\ -4y & -5 \end{bmatrix} = \begin{bmatrix} 10 & 2z \\ 16 & -5 \end{bmatrix}$ $x = 3; y = -4; z = 5$

Find each product.

6. $\begin{bmatrix} 4 & 7 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -5 & 4 \\ 3 & -6 \end{bmatrix}$ $\begin{bmatrix} 1 & -26 \\ 11 & -16 \end{bmatrix}$

7. $\begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}$ $\begin{bmatrix} 13 & 5 \\ -1 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 3 & 6 \end{bmatrix}$ $\begin{bmatrix} 10 & 4 & 19 \end{bmatrix}$

Do you UNDERSTAND?

9. **Writing** Describe the matrix operations that you must use to solve the following matrix equation. Then find the value of X.

$$2\begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} + 2X = \begin{bmatrix} 22 & -2 \\ -2 & 4 \end{bmatrix}$$

First, multiply by the scalar 2. Then, use the Subtraction Property of Equality to isolate the variable matrix. Subtract corresponding elements. Finally, multiply each side by $\frac{1}{2}$ and simplify; $X = \begin{bmatrix} 7 & -2 \\ -3 & 5 \end{bmatrix}$

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Chapter 12 Test (continued) — *Form K*

Do you know HOW?

Determine whether the following matrices are multiplicative inverses.

10. $\begin{bmatrix} -5 & 2 \\ 3 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 3 & 8 \end{bmatrix}$ no

11. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ yes

12. $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1\frac{1}{2} \\ -1 & 2 \end{bmatrix}$ yes

Use inverse matrices to find the solution of each matrix equation.

13. $\begin{bmatrix} 5 & 1 \\ -3 & 8 \end{bmatrix}X = \begin{bmatrix} 54 \\ 2 \end{bmatrix}$ $X = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$

14. $\begin{bmatrix} -3 & 1 \\ 5 & 4 \end{bmatrix}X = \begin{bmatrix} -5 & -17 \\ 14 & 51 \end{bmatrix}$ $X = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$

15. $\begin{bmatrix} 4 & -3 \\ 1 & -8 \end{bmatrix}X = \begin{bmatrix} 14 \\ -11 \end{bmatrix}$ $X = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Use matrices to solve the following systems of equations.

16. $\begin{cases} 4x + 3y = 7 \\ -2x + 4y = 24 \end{cases}$ $x = -2; y = 5$

17. $\begin{cases} x + 20 = -3y \\ 5x = 4 - 2y \end{cases}$ $x = 4; y = -8$

18. $\begin{cases} 4y + 2z = 2 \\ 5z = 20 + 3x \\ 2x + 3z = 7 - 8y \end{cases}$ $x = 5; y = -3; z = 7$

Do you UNDERSTAND?

19. Roger and Clarissa each sold boxes of cookies for a fundraiser. They sold large and small boxes for different prices. Roger sold 12 large boxes and 8 small boxes for a total of $56.00. Clarissa sold 16 large boxes and 11 small boxes for a total of $75.50. Use a system of two equations and matrices to find the price of a large box and a small box. **large: $3.00; small: $2.50**

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Chapter 12 Performance Tasks

Task 1 Check student's work.

a. Discuss the conditions that need to exist so that matrices can be added, subtracted, and multiplied.

b. Write a 3×3 matrix A. Add your matrix to $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

c. Subtract $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ from your original matrix.

d. Multiply your matrix by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What are the dimensions of the product matrix?

e. Solve the matrix equation $C - \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 5 \\ 4 & 2 & 1 \end{bmatrix} = A$, where A is your original matrix.

f. Write a 2×2 matrix. Find the inverse of this matrix. If no inverse exists, explain why.

[4] Discussion shows a clear understanding of matrix operations. Student writes an appropriate matrix. Operations are performed correctly. Inverse is correct or non-existence correctly explained. The only errors are minor computational or copying errors.

[3] Discussion shows some understanding of matrix operations, but may be incomplete. Student writes an appropriate matrix. Operations are performed correctly. Inverse is correct or non-existence correctly explained.

[2] Discussion shows some understanding of matrix operations, but may be incomplete. Student writes an appropriate matrix. Operations show appropriate strategies, but are implemented incorrectly. Inverse is correct or non-existence is at least partly correctly explained.

[1] Discussion shows limited understanding of matrix operations. Student attempts to write an appropriate matrix. Student attempts operations. Inverse is incorrect or incomplete.

[0] Student makes no attempt OR no response is given.

Task 2

Use a matrix equation to solve the system. Show your work. $\begin{cases} -2x + 4y - z = -6 \\ x \quad\quad - z = 1 \\ -x + y \quad\quad = -5 \end{cases}$

[4] Student writes correct matrix equation, and multiplies the inverse of the coefficient matrix by the constant matrix:

$$\begin{bmatrix} -2 & 4 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -4 \\ 1 & -1 & -3 \\ 1 & -2 & -4 \end{bmatrix}\begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 13 \\ 8 \\ 12 \end{bmatrix}$$

[3] Student applies an appropriate solution strategy, with several errors.

[2] Student chooses an appropriate solution strategy, but applies it incorrectly or incompletely. Student shows some understanding of the problem.

[1] Student attempts a solution, but shows little understanding of the problem.

[0] Student makes no attempt OR no response is given.

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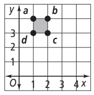
Chapter 12 Performance Tasks (continued)

Task 3



a. Write the vertices of the shaded square as a 2×4 matrix.

b. Translate the square 2 units left and 1 unit down. What are the coordinates of the vertices of the translated square?

c. Dilate the original square by a factor of 2. What are the coordinates of the vertices of the dilated square?

d. Suppose you rotate or reflect the original square. Describe how to produce a matrix representing the coordinates of the vertices of the rotated or reflected square.

[4] Student writes the correct matrix and uses appropriate methods to transform the square. The only errors are minor computational or copying errors: $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 4 & 3 & 3 \end{bmatrix}$; $a'(-1, 3)$, b' (0, 3), c' (0, 2), d' (−1, 2); a' (2, 8), b' (4, 8), c' (4, 6), d' (2, 6); multiply the appropriate rotation or reflection matrix by the original 2×4 matrix. The product matrix represents the coordinates of the vertices of the transformed square.

[3] Student writes correct matrix and uses appropriate methods to transform the square, with several errors. Description of rotation/reflection is mostly correct.

[2] Student writes incorrect matrix, but uses appropriate methods to transform the square OR writes correct matrix, but uses inappropriate methods to transform the square. Description of rotation/reflection is partly correct.

[1] Student attempts the transformations, but shows little understanding of the problem.

[0] Student makes no attempt OR no response is given.

Task 4

Let $P = (25, -25)$, $Q = (-12, 12)$, $R = (-30, -6)$, and $S = (5, 29)$. Use vectors to determine if $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are perpendicular to each other.

[4] Student correctly writes the vectors in component form, $\overrightarrow{PQ} = \langle -37, 37 \rangle$ and $\overrightarrow{RS} = \langle 35, 35 \rangle$, then calculates the dot product, $\overrightarrow{PQ} \cdot \overrightarrow{RS} = (-37)(35) + (37)(35) = 0$. The dot product equals zero. Therefore, the vectors are normal so the lines on which the vectors lie are perpendicular.

[3] Student writes the vectors in component form and calculates the dot product, with minor errors.

[2] Student writes the vectors in component form with minor errors. Student attempts to calculate the dot product, but does so incorrectly or incompletely. Student shows some understanding of the problem.

[1] Student attempts a solution, but shows little understanding of the problem.

[0] Student makes no attempt OR no response is given.

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Chapter 12 Cumulative Review

Multiple Choice

For Exercises 1–12, choose the correct letter.

1. Which line is perpendicular to $y = \frac{3}{4}x - 1$? C
   (A) $y = \frac{4}{3}x - 5$ (B) $y = -\frac{3}{4}x + 2$ (C) $y = -\frac{4}{3}x + 6$ (D) $y = \frac{3}{4}x - 3$

2. Which number is an integer? H
   (F) $\sqrt{6}$ (G) $\frac{3}{5}$ (H) -3 (I) 0.74

3. Which ordered pair is a solution of the system $\begin{cases} x - y \geq 2 \\ x + y \leq 7 \end{cases}$? A
   (A) (5, 1) (B) (−2, 4) (C) (6, 5) (D) (−1, −2)

4. Find $\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$. F
   (F) $\begin{bmatrix} 14 & 5 \\ 14 & 4 \end{bmatrix}$ (G) $\begin{bmatrix} 3 & 2 & 2 & 1 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ (H) $\begin{bmatrix} 14 & 3 \\ 14 & 3 \end{bmatrix}$ (I) $\begin{bmatrix} 14 & 5 \\ 9 & 4 \end{bmatrix}$

5. What is the equation of a line with slope 3 passing through (1, 0)? C
   (A) $y = 3x + 3$ (B) $y = 3x$ (C) $y = 3x - 3$ (D) $y = 3(x - 3)$

6. Which point lies on the graph of $7x - 5y + 2z = 70$? G
   (F) (10, −1, 2) (G) (2, −10, 3) (H) (0, 5, 10) (I) (2, −7, 20)

7. Find the value of 6!. D
   (A) 21 (B) 30 (C) 360 (D) 720

8. Which line contains the point (2, 3)? G
   (F) $y = x - 1$ (G) $y = x + 1$ (H) $y + 1 = x$ (I) $3y = 2x$

9. For which matrix is $a_{21} = 5$? D
   (A) $\begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

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Chapter 12 Cumulative Review (continued)

10. Solve $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}X = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. I
   (F) $\begin{bmatrix} 11 \\ -7 \end{bmatrix}$ (G) $\begin{bmatrix} 2 & \frac{1}{2} \end{bmatrix}$ (H) $\begin{bmatrix} 11 & -7 \end{bmatrix}$ (I) $\begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$

11. At which vertex is the objective function $C = -3x + 7y$ maximized? C
   (A) (0, 1) (B) (5, 2) (C) (−3, 6) (D) (−1, −4)

12. Find $\begin{bmatrix} 5 & 9 \\ 44 & 21 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$. I
   (F) $\begin{bmatrix} 7 & 10 \\ 48 & 21 \end{bmatrix}$ (G) $\begin{bmatrix} 3 & 8 \\ 44 & 0 \end{bmatrix}$ (H) $\begin{bmatrix} 3 & 8 \\ 44 & 0 \end{bmatrix}$ (I) $\begin{bmatrix} 3 & 8 \\ 40 & 21 \end{bmatrix}$

Short Response

13. What is the x-intercept of the line $3x = 4y + 12$? Show your work.
   0; $3x = 4y + 12$; $3x = 4(0) + 12$; $3x = 12$; $x = 4$

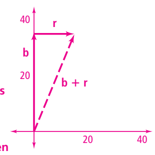
14. The coordinates of the vertices of a triangle are (3, −2), (2, 5), and (−4, 0). Give the coordinates of the vertices of the reflection of the triangle across the line $y = x$. Show your work. (*Hint:* The reflection matrix for a reflection across $y = x$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 & -4 \\ -2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 0 \\ 3 & 2 & -4 \end{bmatrix}$; (−2, 3), (5, 2), (0, −4)

15. A hiker starts at the trailhead, walks 3 mi due east, then turns and walks 4 mi due north. What is the direct distance from the hiker's current position to the trailhead? 5 mi

Extended Response

16. A boat travels 35 mi/h in still water. A current flows due east at 15 mi/h and the boat travels due north across the current.
   a. Draw a sketch with vectors to represent the resultant speed of the boat.
   b. Solve an equation to find the resultant speed of the boat to the nearest integer. Show your work.

[4] $b + r = \langle 15, 35 \rangle$
$|b + r| = \sqrt{15^2 + 35^2} \approx 38$
The resultant speed is about 38 mi/h.



[3] Student uses appropriate strategies, but misunderstood part of the problem or ignored a condition in the problem.

[2] Student attempts to use appropriate strategies, but applies them incorrectly or incompletely.

[1] Student work contains significant errors and little evidence of strategies used.

[0] incorrect answers and no work shown OR no answers given

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Chapter 12 Project Teacher Notes: Munching Microbes

About the Project

The Chapter Project gives students an opportunity to use matrices to organize data from a bioremediation project. They use the matrices to calculate totals and changes in the amount of wastes present. Then, they research other bioremediation projects and summarize and display their findings.

Introducing the Project

Encourage students to keep all project-related materials in a separate folder.

- Ask students if they have been near sites that were being cleaned up after oil or chemical spills. Explain that the field of bioremediation uses naturally-occurring bacteria to degrade hazardous wastes.
- Have students make a list of materials they need to begin the project.

Activity 1: Organizing

Students write matrices with data from soil tests for hazardous wastes.

Activity 2: Calculating

Students use their previously-written matrices to calculate the total amounts of the hazardous waste components and to show by how much the amount of each component has decreased over 12 months.

Activity 3: Researching

Students research a hazardous waste clean-up that includes bioremediation. They then write a report on the clean-up.

Finishing the Project

You may wish to plan a project day on which students share their completed projects. Encourage students to explain their processes as well as their results. Ask students to review their project work and update their folders.

- Have students review their methods for organizing the data in matrices, calculating totals and changes in the component amounts, and for researching bioremediation for the project.
- Ask groups to share their insights that resulted from completing the project, such as any shortcuts they found for organizing the data or conducting research.

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Chapter 12 Project: Munching Microbes

Beginning the Chapter Project

Oil spills and chemical contamination of groundwater are some of the present-day hazards. The field of bioremediation uses bacteria that occur naturally in the environment to decompose hazardous wastes.

In this project, you will organize data from a bioremediation project. You will manipulate the data and use the results to draw conclusions and make predictions. Then you will research other bioremediation projects. Finally, you will summarize and display your findings.

List of Materials

- Calculator
- Graph paper
- Poster board

Activities

Activity 1: Organizing

The table shows data from an above-ground biotreatment project. Scientists analyzed five samples from the same soil for the presence of hazardous components of petroleum products. They found benzene (B), toluene (T), ethylbenzene (E), nd xylene (X).

Component Levels in Soil (mg/kg)

Sample	B	T	E	X
1	0.06	0.95	0.9	18.5
2	0.06	1.05	0.73	13.5
3	0.35	6	5.6	49
4	0.22	0.19	2	19.5
5	0.11	0.82	2.5	26

- Present the data in four matrices.
- Choose an element from each matrix and tell what it represents.

$$\begin{array}{cc} & B \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0.06\\0.06\\0.35\\0.22\\0.11\end{bmatrix}\end{array}, \begin{array}{cc} & T \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0.95\\1.05\\6\\0.19\\0.82\end{bmatrix}\end{array}, \begin{array}{cc} & E \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0.9\\0.73\\5.6\\2\\2.5\end{bmatrix}\end{array}, \begin{array}{cc} & X \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}18.5\\13.5\\49\\19.5\\26\end{bmatrix}\end{array}$$; Check students' work.

Activity 2: Calculating

In this activity, you will use the matrices you wrote for Activity 1.

- Use matrices to find the combined amount of benzene, toluene, ethylbenzene, and xylene in mg/kg for each soil sample.
- After 12 months of bioremediation, the levels of each component dropped to less than 0.05 mg/kg for each soil sample. Use matrices to calculate the minimum amount by which the level of each component for each sample dropped.

combined amounts (mg/kg): 20.41, 15.34, 60.95, 21.91, 29.43

$$\begin{array}{cc} & B \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0.01\\0.01\\0.30\\0.17\\0.06\end{bmatrix}\end{array}, \begin{array}{cc} & T \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0.90\\1.00\\5.95\\0.14\\0.77\end{bmatrix}\end{array}, \begin{array}{cc} & E \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0.85\\0.68\\5.55\\1.95\\2.45\end{bmatrix}\end{array}, \begin{array}{cc} & X \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}18.45\\13.45\\48.95\\19.45\\25.95\end{bmatrix}\end{array}$$

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Chapter 12 Project: Munching Microbes (continued)

Activity 3: Researching

Research a hazardous waste clean-up that includes bioremediation. How large is the site? What treatment methods other than bioremediation are being used, if any? Write a few paragraphs summarizing your research. Include data from the site, if possible. Check students' work.

Finishing the Project

The activities should help you complete your project. You should prepare a presentation for the class describing some aspect of a bioremediation project. Your presentation could be a graph or chart analyzing bioremediation data or a poster advertising a bioremediation project.

Reflect and Revise

Ask a classmate to review your project with you. After you have seen each other's presentations, decide if your work is complete, clear, and convincing. Make sure that you have included all supporting material from your work on the project. Check that the information you have presented is accurate. You may want to make some changes based on your classmate's review.

Extending the Project

Bioremediation is a fairly new field. You can do more research into the field by contacting the United States Department of the Interior. You can also get more information on the Internet.

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Chapter 12 Project Manager: Munching Microbes

Getting Started

Read the project. As you work on the project, you will need a calculator and materials on which you can record your calculations and make neat matrices. Keep all of your work for the project in a folder.

Checklist	Suggestions
☐ Activity 1: organizing data with matrices	☐ Arrange amounts of each component in a 5×1 matrix.
☐ Activity 2: using matrix operations	☐ Add corresponding entries to find the totals.
☐ Activity 3: researching bioremediation	☐ Check the Internet for information.
☐ bioremediation report	☐ How has hazardous waste clean-up changed in the last twenty years? What standards exist by which it is determined that a site is considered hazardous or that a site has been sufficiently cleaned up? Do these standards vary by state?

Scoring Rubric

4 Calculations are correct. Matrices are neat and accurate. Explanations are thorough and well thought out. The presentation is accurate and thoroughly explains the information.

3 Calculations are mostly correct, with some minor errors. Matrices are neat and mostly accurate with minor errors. The explanations lack detail. The presentation has minor errors or lacks important information.

2 Calculations contain both minor and major errors. Matrices are not accurate. The explanations and presentation are inaccurate or incomplete.

1 Major elements of the project are incomplete or missing.

0 Project is not turned in or shows no effort.

Your Evaluation of Project Evaluate your work, based on the *Scoring Rubric.*

Teacher's Evaluation of the Project