Polar Coordinates and Functions Key Concepts

Polar Coordinates Lesson

Polar Coordinates

Point $P(r, \theta)$ is located a directed distance, r, from the pole at an angle of rotation, θ, from the polar axis.

Polar to Rectangular Coordinates

Point $P(r, \theta)$ can be converted to rectangular coordinates (x, y) using the following formulas:

$x = r \cos \theta$

$y = r \sin \theta$

Rectangular to Polar Coordinates

Point $P(x, y)$ can be converted to polar coordinates (r, θ) using the following formulas:

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$ and $r = \sqrt{x^2 + y^2}$

Polar Equations Lesson

Converting Equations Between Polar and Rectangular Forms

The following formulas can be used to convert equations between polar and rectangular forms:

$x = r \cos \theta$

$y = r \sin \theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$

$r = \sqrt{x^2 + y^2}$

Graphs of Polar Equations Lesson

Circles in Polar Form

The graphs of $r = a \cos \theta$ and $r = a \sin \theta$ are circles where a is the diameter of the circle.

If $a < 0$, then $r = a \cos \theta$ will be on and left of the pole and $r = a \sin \theta$ will be on and below the pole.

If $a > 0$, then $r = a \cos \theta$ will be on and right of the pole and $r = a \sin \theta$ will be on and above the pole.

Tests for Symmetry in Polar Coordinates

A curve is symmetric about the polar axis if both (r, θ) and $(r, -\theta)$, or both (r, θ) and $(-r, \pi - \theta)$ are on the curve.

A curve is symmetric about the vertical axis if both (r, θ) and $(-r, -\theta)$, or both (r, θ) and $(r, \pi - \theta)$ are on the curve.

A curve is symmetric about the pole if both (r, θ) and $(-r, \theta)$, or both (r, θ) and $(r, \pi + \theta)$ are on the curve.

Conic Sections in Polar Coordinates Lesson

Polar Equations of Conic Sections

The graph of a polar equation in the form of $r = \frac{ed}{1 \pm e \cos \theta}$ or $r = \frac{ed}{1 \pm e \sin \theta}$ is a conic section, where e is the eccentricity and $|d|$ is the distance between the focus (pole) and the directrix.

If $0 < e < 1$, then the conic section is an ellipse.

If $e = 1$, then the conic section is a parabola.
If $e > 1$, then the conic section is a hyperbola.

Limaçons Lesson

Limaçons

The standard equations for graphs of limaçons are $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$, where $a > 0$ and $b > 0$.

If $\frac{a}{b} < 1$, the graph is an inner loop limaçon.

If $\frac{a}{b} = 1$, the graph is a cardioid.

If $1 < \frac{a}{b} < 2$, the graph is a dimpled limaçon with no inner loop.

If $\frac{a}{b} \geq 2$, the graph is a convex limaçon (no dimple and no inner loop).

Cardioids

The graphs of cardioids are represented by the equations $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, where *a* = *b*, $\frac{a}{b} = 1$, $a > 0$, and $b > 0$.

equation	$r = a + b\cos\theta$	$r = a - b\cos\theta$	$r = a + b\sin\theta$	$r = a - b\sin\theta$
axis of symmetry	horizontal	horizontal	vertical	vertical
placement	lies on and mostly right of the pole	lies on and mostly left of the pole	lies on and mostly above the pole	lies on and mostly below the pole
horizontal intercepts	0 and $2a$	0 and $-2a$	a and $-a$	a and $-a$
vertical intercepts	a and $-a$	a and $-a$	0 and $2a$	0 and $-2a$

Dimpled and Convex Limaçons

Dimpled: The graphs of dimpled limaçons are represented by the equations $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, where the ratio of *a* to *b* is $1 < \frac{a}{b} < 2$, $a > 0$, and $b > 0$.

Convex**:** The graphs of convex limaçons are represented by the equations $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, where the ratio of *a* to *b* is $\frac{a}{b} \geq 2$, $a > 0$, and $b > 0$.

equation	$r = a + b\cos\theta$	$r = a - b\cos\theta$	$r = a + b\sin\theta$	$r = a - b\sin\theta$
axis of symmetry	horizontal	horizontal	vertical	vertical

placement	lies mostly right of the pole	lies mostly left of the pole	lies mostly above the pole	lies mostly below the pole
horizontal intercepts	*a* + *b* units to the right of the pole and $a-b$ units to the left of the pole	*a* + *b* units to the left of the pole and $a-b$ units to the right of the pole	*a* and $-a$	*a* and $-a$
vertical intercepts	*a* and $-a$	*a* and $-a$	*a* + *b* units above the pole and $a-b$ units below the pole	*a* + *b* units below the pole and $a-b$ units above the pole

Inner Loop Limaçons

The graphs of inner loop limaçons $r=a\pm b\cos\theta$ and $r=a\pm b\sin\theta$, where the ratio of *a* to *b* is $\frac{a}{b}<1$, $a>0$, and $b>0$.

equation	$r = a + b \cos \theta$	$r=a-b\cos\theta$	$r = a + b \sin \theta$	$r=a-b\sin\theta$
axis of symmetry	horizontal	horizontal	vertical	vertical
placement	lies on and mostly right of the pole	lies on and mostly left of the pole	lies on and mostly above the pole	lies on and mostly below the pole
horizontal intercepts	0, *a* + *b* and $b-a$ units to the right of the pole	0, *a* + *b* and $b-a$ units to the left of the pole	*a* and $-a$	*a* and $-a$
vertical intercepts	*a* and $-a$	*a* and $-a$	0, *a* + *b*, and $b-a$ units above the pole	0, *a* + *b*, and $b-a$ units below the pole

Rose Curves and Lemniscates Lesson

Rose Curves

The standard equations for graphs of rose curves are $r = a \sin n\theta$ and $r = a \cos n\theta$, where $a\neq 0$ and *n* is an integer greater than 1.

The length of the petals (also known as loops) is determined by the value of *a*.

The number of petals is determined by the value of *n*. If *n* is even, there are 2*n* number of petals/loops. If *n* is odd then there are *n* number of petals/loops.

Lemniscates

The standard equations for graphs of lemniscates are $r^2 = a^2 \sin 2\theta$ and $r^2 = a^2 \cos 2\theta$ where $a \neq 0$.

A lemniscate has the shape of a figure eight or an airplane propeller centered at the pole.

The total distance from the center to the end of each loop is equal to *a*.

The lemniscate $r^2 = a^2 \sin 2\theta$, where $a \neq 0$, is symmetric with respect to the pole.

The graph of $r^2 = -a^2 \sin 2\theta$ is a reflection across the vertical axis of $r^2 = a^2 \sin 2\theta$.

The lemniscate $r^2 = a^2 \cos 2\theta$, where $a \neq 0$, is symmetric with respect to both the horizontal axis and the pole.

The graph $r^2 = -a^2 \cos 2\theta$ is a reflection across the line $\theta = \frac{\pi}{4}$ of $r^2 = a^2 \cos 2\theta$.

Complex Numbers in Polar Form Lesson

Rectangular Form of a Complex Number

The rectangular form of a complex number is *z* = *a* + *bi*, where the following applies:

- *a* is called the real part.
- *b* is called the imaginary part.
- $|z| = \sqrt{a^2 + b^2}$ is the absolute value of a complex number, or the modulus.

Polar Form of a Complex Number

The polar form of the complex number is $z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$, where the following applies:

- *a* = *r* cos *θ* and *b* = *r* sin *θ*
- $\tan\theta = \frac{b}{a}, \ \ a \neq 0$

- $\cos\theta = \frac{a}{r}$
- $\sin\theta = \frac{b}{r}$
- The number *r* is the modulus of *z*.
- *θ* is called the argument of *z*.
- $r = |z|$

The representation is unique for $0 \le \theta \le 2\pi$ for every *z* except 0 + 0*i*.

Operations of Complex Numbers in Polar Form Lesson

Multiplying Complex Numbers in Polar Form

Let $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$.

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Dividing Complex Numbers in Polar Form

Let $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

DeMoivre's Theorem

DeMoivre's Theorem

- Let $z = r(\cos\theta + i\sin\theta)$ be a complex number, where *n* is an integer, $n \ge 1$.
- If $z^n = r^n(\cos\theta + i\sin\theta)^n$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

*n*th Root of a Complex Number

If *n* is any positive integer, the *n*th roots of *z* = *r* cis *θ* are given by

$\sqrt[n]{r \operatorname{cis} \theta} = (r \operatorname{cis} \theta)^{\frac{1}{n}}$, where the *n* roots are found with the formula:

- $\sqrt[n]{r}\,\text{cis}\left(\dfrac{\theta+360^\circ k}{n}\right)$ for degrees
- $\sqrt[n]{r}\,\text{cis}\left(\dfrac{\theta+2\pi k}{n}\right)$ for radians

for $k=0,1,2,\ldots,n-1$