UNIT: SEQUENCES & SERIES

SEQUENCES

A sequence is a list of numbers that follows an orderly pattern.

The pattern can be written as a rule or formula.

TYPES OF SEQUENCES

The two types of sequences we are mainly working with are the Arithmetic sequences and the Geometric sequences.

But there are many others. For fun, go search the Fibonacci sequence and the many items in nature that follow the spiral that comes from this sequence.

TYPES OF PATTERN RULES

A Recursive rule has two parts: the recursive pattern, and the naming of the first term.

A Recursive rule describes the pattern in terms of the previous term. For example, each term is two more than the previous term, and the sequence starts with 1. This would be the sequence: 1, 3, 5, 7, …

An Explicit rule describes the pattern in terms of the position number (that is, 1^{st} number in the list, 2^{nd} number on the list, and so forth). For example, each term is three times its position number. So the 1^{st} term would be 1x3, the 2^{nd} term would be 2x3, and so forth. This sequence would be: 3, 6, 9, 12, …

NOTATION FOR SEQUENCES

a_n is the term in position n, or the "n^{th} " term

a_1 is the 1^{st} term, a_2 is the 2^{nd} term, and so forth

a_{n-1} is the previous term, a_{n+1} is the next term

n is the term number, or, the position in the list (that is 1 for 1^{st} , 2 for 2^{nd}, …)

ARITHMETIC SEQUENCE

Definition: an ordered list of numbers with a constant difference, that is, the same number gets added to each term to get the next term on the list.

d is the amount of the constant difference

RECURSIVE RULE for an ARITHMETIC SEQUENCE

$a_n = a_{n-1} + d$; a_1 = a

For example, the sequence 1, 3, 5, 7, … has a constant difference of 2 and the first term is the number 1, so it's recursive rule would be:

$a_n = a_{n-1} + 2$; $a_1 = 1$

EXPLICIT RULE for an ARITHMETIC SEQUENCE

$a_n = a_1$ + (n-1)d

For example, the same sequence 1, 3, 5, 7, … has the explicit rule of:

a_n = 1 + (n-1)2

After distributing and simplifying, this rule can be also written as:

a_n = 2n – 1

ARITHMETIC MEAN

This is for finding the number between two terms of an Arithmetic sequence.

$$\frac{x+y}{2}$$

For example, the number between 3 and 7 in the Arithmetic sequence above is $\frac{3+7}{2} = 5$.

GEOMETRIC SEQUENCE

Definition: an ordered list of numbers with a constant ratio, that is, the same number gets multiplied by each term to get the next term in the list.

r is the amount of the constant ratio

For example, the sequence 2, 6, 18, 54, … has a constant ratio of 3, that is, each term is three times the one before.

RECURSIVE RULE for a GEOMETRIC SEQUENCE

$a_n = a_{n-1} \cdot r$; a_1 = a

For example, the geometric sequence above would have a recursive rule of:

$a_n = a_{n-1} \cdot 3$; a_1 = 2

because the constant ratio is 3, and the first term is 2.

EXPLICIT RULE for a GEOMETRIC SEQUENCE

$a_n = a_1 \cdot r^{n-1}$

For example, the geometric sequence above would have an explicit rule of:

$a_n = 2 \cdot 3^{n-1}$

Because the first term is 2, and the constant ratio is 3.

GEOMETRIC MEAN

This is for finding the number between two terms of a Geometric sequence.

$\sqrt{xy}$

For example, the number between 2 and 18 in the Geometric sequence above is

$\sqrt{2 \cdot 18} = \sqrt{36} = 6.$

SERIES

A Series is the sum of the terms of a Sequence.

So, essentially, just replace the commas with addition signs.

TYPES OF SERIES

Finite series have a limited number of terms.

For example, 1 + 3 + 5 + 7 is a finite series.

Infinite series have an endless number of terms.

For example, 1 + 3 + 5 + 7 + 9 + … is an infinite series.

SUMMATION (SIGMA) NOTATION

This is the shorthand way for writing the pattern for a sequence and including which terms of the list are being added in the series.

Sigma ($\sum$) is the Greek letter for capital S, and is used to mean a sum.

This notation can be used for describing any type of sequence that is being added as a series.

Below the $\sum$ is written the n value for the first position to be added in the series, and above the $\sum$ is written the n value for the last position to be added in the series.

To the right of the $\sum$ is written the Explicit rule for the sequence being used in the summation.

For example,

$$\sum_{n=1}^{5} 2n - 1$$

is the summation notation for adding the first 5 odd numbers.

So, this means to start with plugging in 1 for n in 2n-1 to get the first term of the sequence. Then plug in 2 to get the second term, and so forth to get all five terms indicated. Then add these five terms to get the sum of this finite series.

SUM of a FINITE ARITHMETIC SERIES

S_n is the sum of the first n terms

n is the number of terms to be added

a_1 is the first number in the series

a_n is the last number to be added in the series

The formula for the sum of a finite arithmetic series is:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

For example, the sum of the first five odd numbers is:

$$S_5 = \frac{5}{2}(1+9) = 25$$

because the first odd number is 1, the fifth odd number is 9, and there are 5 numbers being added.

FINDING "n"

**Sometimes you have a long series, and you don't know the value for n (position number) for the last term of the series.

You may need this for the upper number for a summation (sigma) notation, or for using the formula to find the sum.

For example, if the series to be summed is 3 + 6 + 9 + … + 78 + 81. We see that this is an arithmetic series with a constant difference of 3, but it's not clear how many terms are in the list.

Start with the explicit rule, which for this would be $a_n = 3n$. To find the value of n for the last term, 81, we plug this last term into the rule and solve for n to find its position number.

So, if $a_n = 3n$, then for a_n = 81, we have 81 = 3n, or n = 27. This means 81 is the 27^{th} term in this list. We can now use n in the formula to find the sum of the series.

This process works for geometric series, as well.

SUM of a FINITE GEOMETRIC SERIES

r is the constant ratio of the terms in the series

The formula for the sum of a finite geometric series is:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

For example, the sum of the finite geometric series 2 + 6 + 18 + 54 has the first term of 2, a constant ratio of 3, and an n value of 4 because there are four terms to be added.

So, the sum of this series is: $S_4 = \frac{2(1-3^4)}{1-3} = \frac{2(1-81)}{-2} = 80$.

DIVERGENT INFINITE SERIES

The sum of most infinite series is undefined. It doesn't make sense to have a final sum when you keep adding another number.

CONVERGENT INFINITE SERIES

However, with some geometric series, if the numbers you keep adding are getting smaller, you can say that the sum of that infinite series does eventually converge on a single value over time.

An infinite geometric series will converge on a sum only if the constant ratio is between 1 and -1 (that is, $-1 < r < 1$, which could also be written as $|r|<1$). Any r value outside of that range will make a series that diverges and will not have a defined sum.

If the series meets the criteria for converging, that is the $|r| < 1$, then the formula for the sum is:

$$S = \frac{a_1}{1-r} \text{ ; for } |r| < 1$$

For example, the geometric series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ has a constant ratio of ½, which meets the criteria for a converging series, and has a first term of 1.

So, the series converges on the sum: $S = \frac{1}{1-\frac{1}{2}} = 2$.

Summary:

SEQUENCES

a_n is the term in position n, or the "n^{th} " term

a_1 is the first term, a_2 is the second term, a_{n-1} is the previous term

n is the term number, or, the position in the list

d is the amount of the constant difference in an arithmetic sequence

r is the amount of the constant ratio in a geometric sequence

Recursive Rule for an Arithmetic Sequence: $a_n = a_{n-1} + d$; $a_1 = a$

Explicit Rule For An Arithmetic Sequence: $a_n = a_1 + d(n-1)$

Recursive Rule For A Geometric Sequence: $a_n = a_{n-1} \cdot r$; $a_1 = a$

Explicit Rule For A Geometric Sequence: $a_n = a_1 \cdot r^{n-1}$

Arithmetic Mean: $\frac{x+y}{2}$, for the number between two terms of an arithmetic sequence.

Geometric Mean: $\sqrt{xy}$, for the number between two terms of a geometric sequence.

SERIES

S_n is the sum of the first n terms

n is the number of terms to be added

a_1 is the first number in the series, a_n is the last number to be added in the series

Sum of a finite arithmetic series: $S_n = \frac{n}{2}(a_1 + a_n)$

Sum of a finite geometric series: $S_n = \frac{a_1(1-r^n)}{1-r}$

Sum of a converging infinite geometric series: $S = \frac{a_1}{1-r}$; for $|r| < 1$

Summation Notation

$$\sum_{n=1}^{last\ n\ position\ to\ be\ added} \quad explicit\ rule\ for\ the\ sequence$$